Reliability of Structures – Part 2

Andrzej S. Nowak
Auburn University

Outline

- Part 1 Uncertainty in structural engineering
  Random variables
- Part 2 Limit States
  Reliability analysis procedures
- Part 3 Code development
  Load and resistance models
- Part 4 System reliability
  Current research and future trends

Structural Safety Analysis

- Limit State
- Fundamental Case
- Reliability Index
- Hasofer-Lind Reliability Index
- Rackwitz-Fiessler Procedure
- Monte Carlo Simulations
- Summary of Reliability Analysis Procedures

Safe Performance and Failure

It is assumed that a structure can be in one of two states:

- Safe when it is able to perform its function
- Unsafe when it cannot perform its function

Inability to perform is a failure

Function requires a clear definition as it can be very subjective

Examples of Definition of Failure

- When load exceeds load carrying capacity
- When deflection exceeds the maximum allowed deflection
- When stress in steel reaches yield stress
- When a column buckles
- When a local buckling occurs in a flange or web of a steel beam
- When there is cracking in concrete
- When there is excessive vibration

Some of these definitions are very subjective

Examples of Failure

- A steel beam may fail by developing a plastic hinge, loss of overall stability or by local buckling

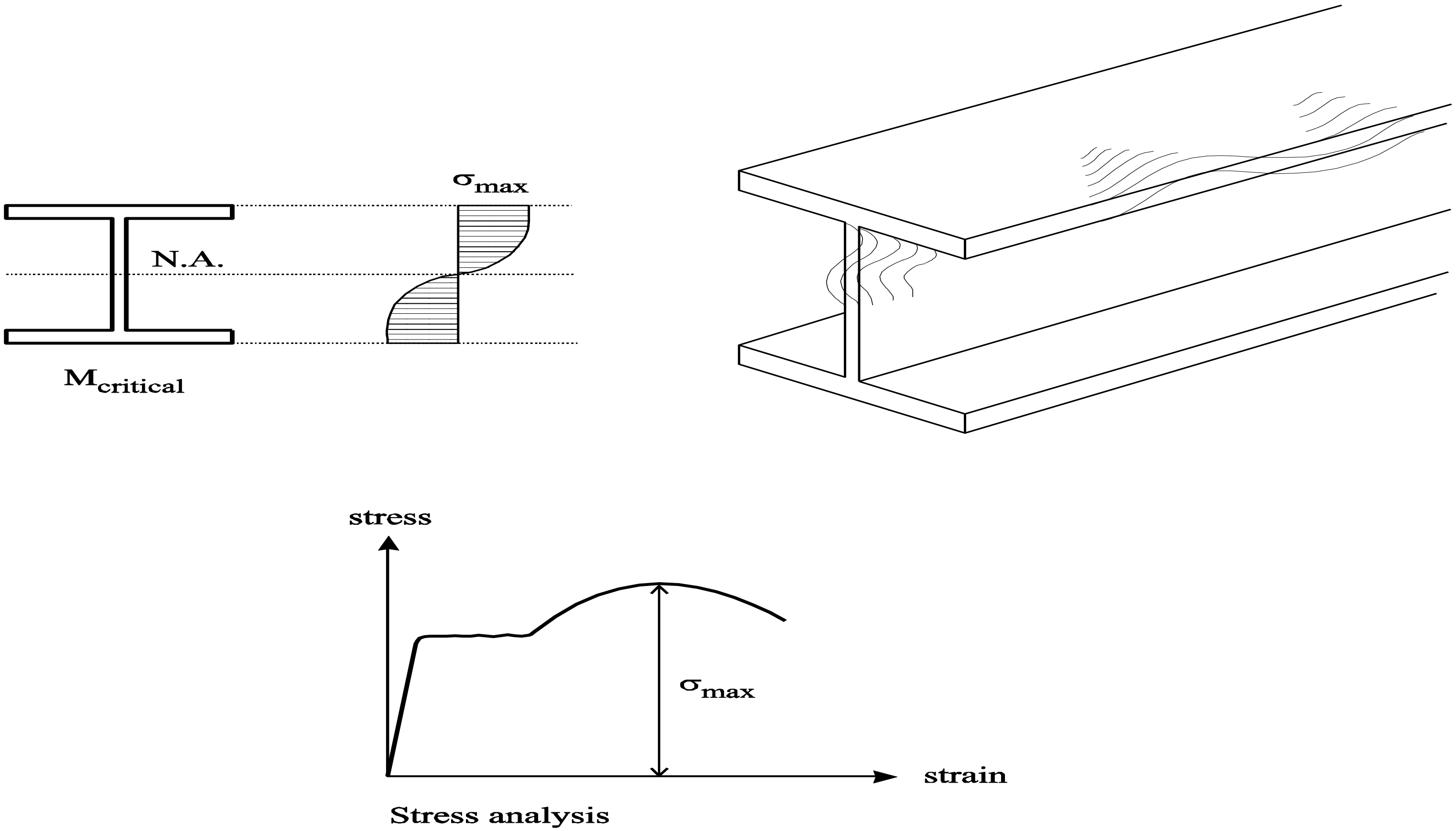


Figure 5-3 Local buckling in a steel beam.

Example of Failure

- Failure must be clearly defined
- Example: Consider a simply supported, hot-rolled, steel beam

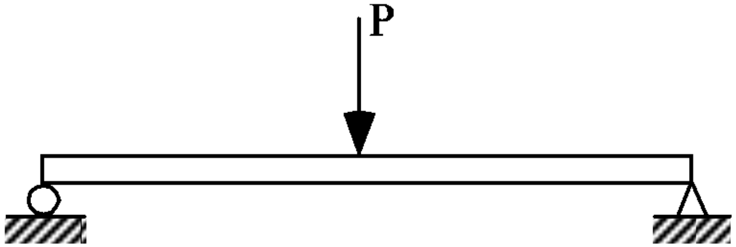


Figure 5-1 A simply supported beam

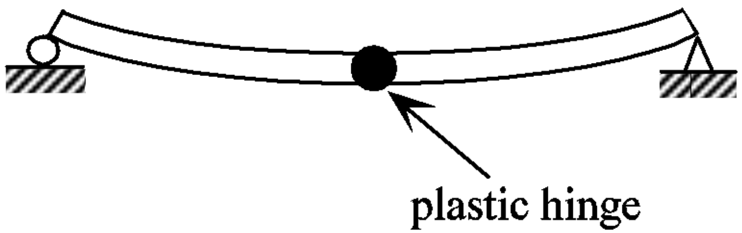


Figure 5-2 Development of a plastic hinge in beam.

- The beam fails when deflection exceeds $\delta_{critical}$

Limit State

In each of these cases, there is a critical safe state on the borderline between safety and failure. This borderline state is called a limit state.

Mathematical representation of a limit state is a limit state function.

Limit state function can be simple or very complex, depending on load and resistance parameters, dimensions, time and so on.

Example of a Limit State Function

Let R be the moment carrying capacity (or resistance) and Q be the load effect. Then, the limit state function can be formulated as

$$g = R - Q = 0$$

If $g < 0$ or $R < Q$, the limit state is exceeded and the structure fails, otherwise the structure is safe.

Probability of Failure

Then the probability of failure, P_F,

$$P_F = P(g < 0) = P(R < Q)$$

Limit State Function for a Steel Beam

The limit state function for a compact steel beam

$$g = Z_x F_y - D - L - W = 0$$

where Z_x = plastic section modulus, F_y = yield stress, D = dead load, L = live load, W = wind load

If $g < 0$, the limit state is exceeded and the structure fails, otherwise the structure is safe.

Types of Limit States

Four limit states are considered:

- Ultimate Limit States, ULS
- Serviceability Limit States, SLS
- Fatigue Limit States
- Extreme Events Limit States

Ultimate Limit States

Related to loss of the load carrying capacity

- Exceeding the moment carrying capacity
- Formation of a plastic hinge
- Crushing of concrete in compression
- Shear failure of the web in a steel beam
- Loss of the overall stability
- Buckling of flange/web
- Weld rupture

Serviceability Limit States

Related to performance of the function

- Cracking of concrete
- Deflection
- Vibration
- Permanent Deformations

Serviceability Limit States

Cracking

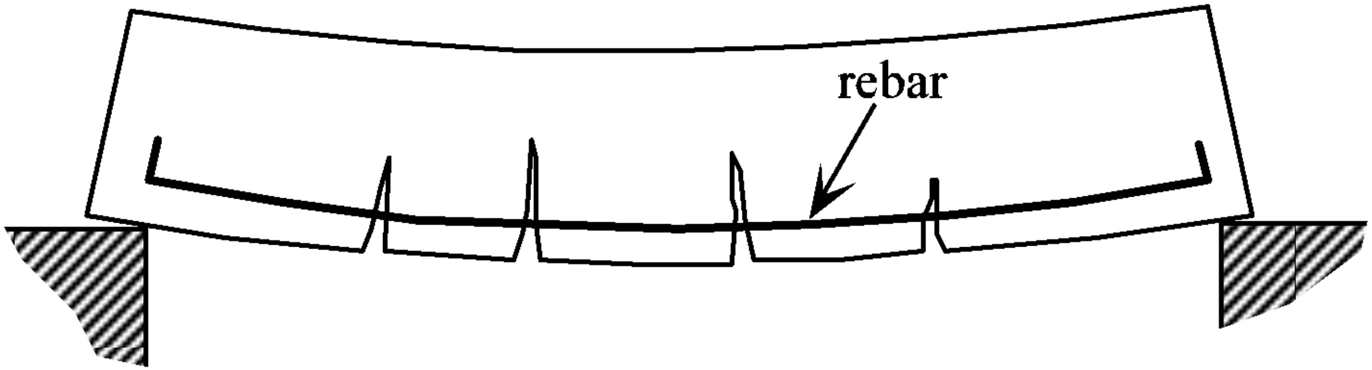


Figure 5-7 Cracks in a reinforced concrete beam.

What is acceptable with regard to cracking? Are acceptable cracks limited by size? Length? Width?

Serviceability Limit States

Deflection

- The acceptable limits are subjective, depend on human perception.
- A building with a visible deflection is not acceptable, even though the structure can be structurally safe.
- For a bridge, limit deflection due to live load is L/800, where L is the span length.

Serviceability Limit States

Vibration

- Difficult to quantify
- In a building, the occupants cannot tolerate an excessive vibration
- For bridges, shaking can be tolerated if no pedestrians are involved

Serviceability Limit States

Important questions:

- What is acceptable vibration/deflection?
- How frequently can those limits states be exceeded?
- How to measure vibrations?

Serviceability Limit States

Permanent deformation

- Accumulation of permanent deflection can lead to serviceability problems

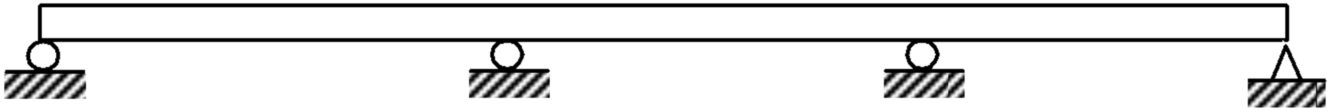


Figure 5-4 Continuous Bridge Girder.

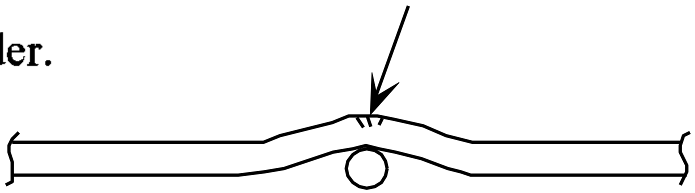


Figure 5-6 Formation of a kink in a continuous steel beam.

Fatigue Limit States

- Related to the accumulation of damage and eventual failure under repeated loads
- Structural member can fail under loads at the level lower the ultimate load
  - Formulation of fatigue limit state for structural steel and reinforced concrete
  - Acceptability criteria
  - Practical design and evaluation criteria

Extreme Events Limit States

- Earthquake
- Collision (vehicle or vessel)
- Flood
- Hurricane
- Tornado
- Act of terrorism

Limit State Function

All realizations of a structure can be put into one of the two categories

- Safe (load effect ≤ resistance)
- Failure (load effect > resistance)

Each limit state is associated with a certain limit state function.

Limit State Function

- The state of the structure can be described using parameters, $X_1, \ldots, X_n$ where X_i's are load and resistance parameters

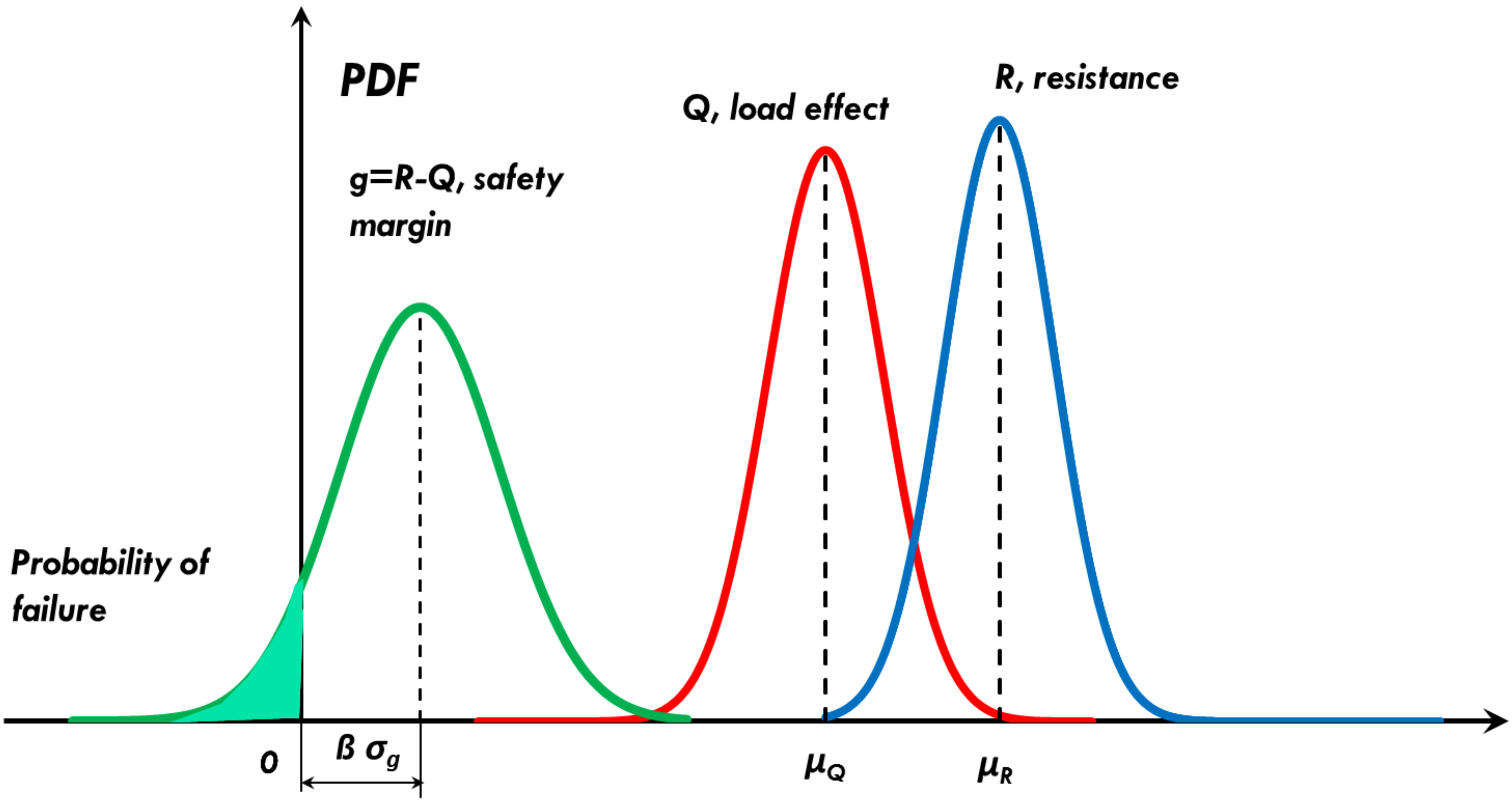
- A limit state function is a function $g(X_1,\ldots,X_n)$ of these parameters, such that

$$g(X_1,\ldots,X_n) \geq 0 \text{ for a safe realization}$$

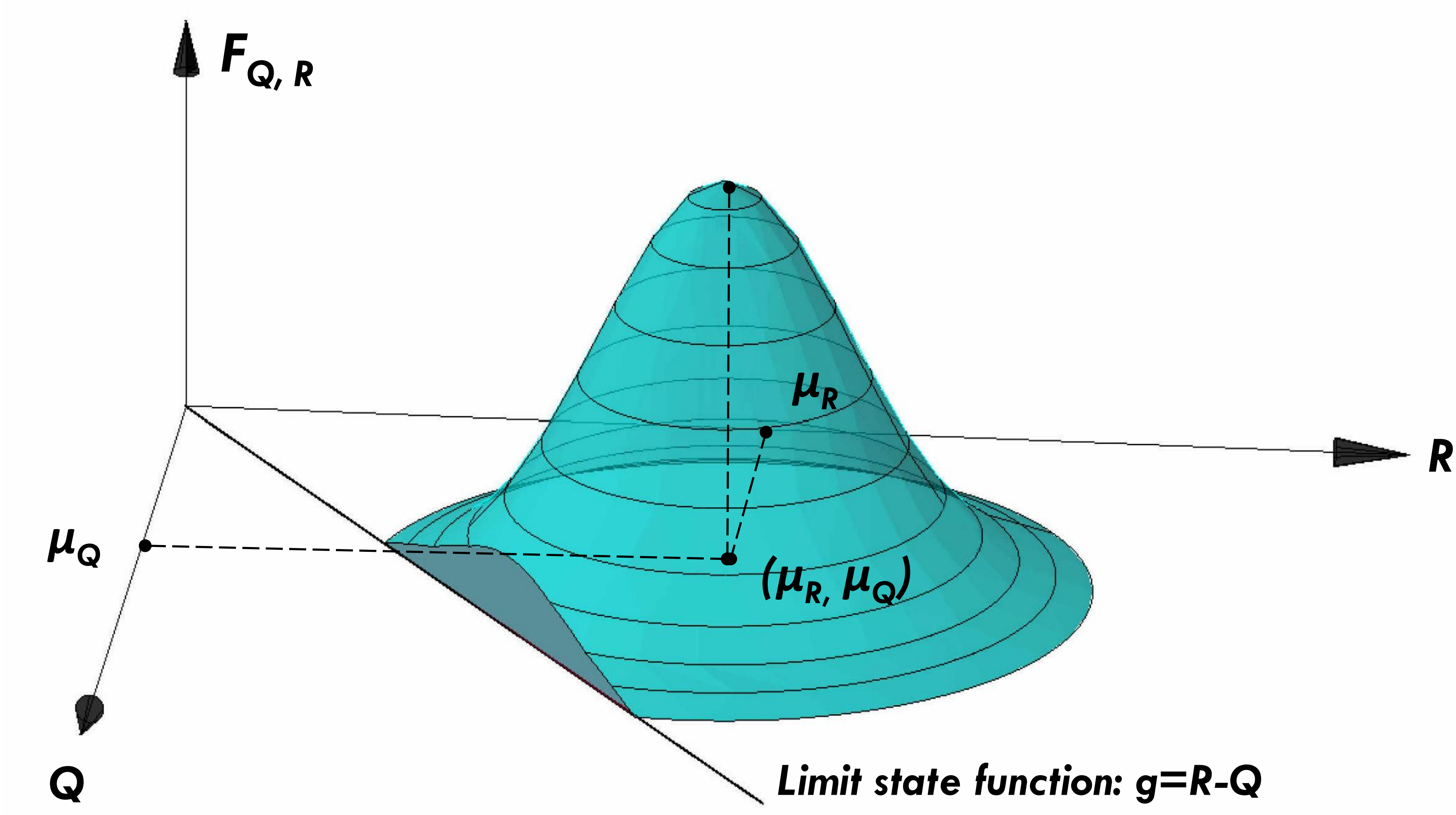
$$g(X_1,\ldots,X_n) < 0 \text{ for failure}$$

Probability of Failure and Reliability Index β

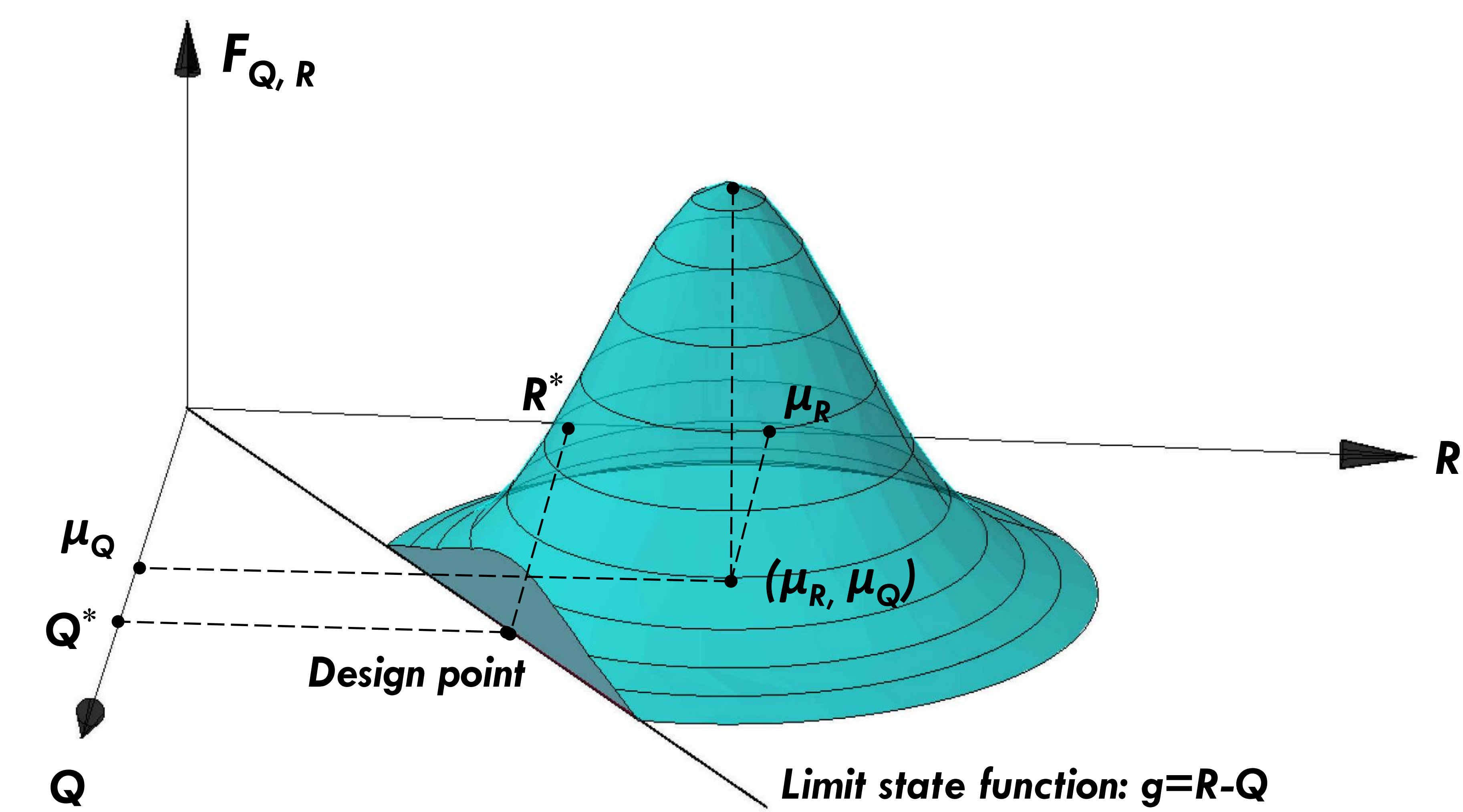
Q = load and R = resistance



Fundamental case

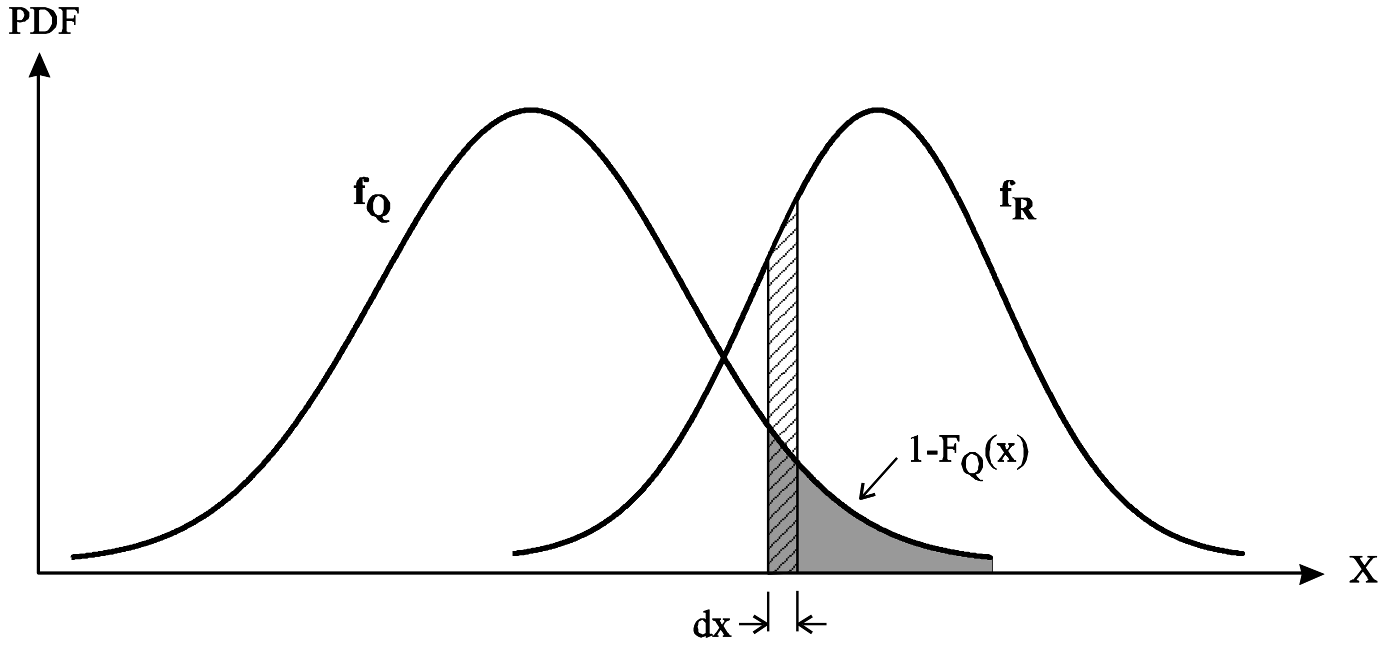


Fundamental case



Fundamental case

From the previous limit state function, g = R - Q, the probability of failure, P_F, can be derived considering the PDF's of R and Q



Fundamental case

The structure fails when the load exceeds the resistance, then the probability of failure is equal to the probability of Q>R, the following equations result

$$P_f = \sum P(R = r_i \cap Q > r_i) = \sum P(Q > R \mid R = r_i) P(R = r_i)$$

$$P_f = \int_{-\infty}^{+\infty} \left(1 - F_Q(r_i)\right) f_R(r_i) dr_i = 1 - \int_{-\infty}^{+\infty} F_Q(r_i) f_R(r_i) dr_i$$

$$P_f = \sum P(Q = q_i \cap R < q_i) = \sum P(R < Q \mid Q = q_i) P(Q = q_i)$$

$$P_f = \int_{-\infty}^{+\infty} F_R(q_i) f_Q(q_i) dq_i$$

Too difficult to use, therefore, other procedures are used

Cornell Reliability Index

Depending on complexity of g function, it can be very difficult to calculate the probability of failure, P_F,

$$P_F = P(g < 0)$$

Cornell (1968) proposed to measure reliability in terms of the reliability index, β,

$$\beta = \mu_g / \sigma_g$$

where μ_g = mean of g and σ_g = standard deviation of g

Reliability Index, β

If the limit state function g = R – Q, where R and Q are independent random variables, then

$$\mu_g = \mu_g - \mu_Q$$

$$\sigma_g{}^2 = \sigma_R{}^2 + \sigma_Q{}^2$$

where μ_R = mean resistance, μ_Q = mean load, σ_R = standard deviation of resistance, σ_Q = standard deviation of load

Reliability Index, β

The reliability index can be calculated using the following formula (Cornell 1968)

$$\beta = \frac{\mu_R - \mu_Q}{\sqrt{\sigma_R^2 + \sigma_Q^2}}$$

where μ_R = mean resistance, μ_Q = mean load, σ_R = standard deviation of resistance, σ_Q = standard deviation of load

Reliability Index, β

If R and Q are independent normal random variables then $g = R - Q$ is also a normal random variable and

$$P_F = P(g<0) = \Phi[(0- \mu_g)/\sigma_g]$$

$$P_F = \Phi(- \mu_g/\sigma_g) = \Phi(-\beta)$$

Reliability Index, β

If R and Q are independent normal random variables then the reliability index calculated using Cornell's formula (Cornell 1968)

$$\beta = \frac{\mu_R - \mu_Q}{\sqrt{\sigma_R^2 + \sigma_Q^2}}$$

is related to the probability of failure as follows,

$\beta = -\Phi^{-1}(Pf)$ or $P_f = \Phi(-\beta)$

Reliability Index

- Relationship between β and P_f

P_f	β
10^{-1}	1.28
10^{-2}	2.33
10^{-3}	3.09
10^{-4}	3.71
10^{-5}	4.26
10^{-6}	4.75
10^{-7}	5.19
10^{-8}	5.62
10^{-9}	5.99

Reliability Index n-Dimensional Case

- Let's consider a linear limit state function

$$g\,(X_1, X_2, \ldots, X_n) = a_0 + a_1 X_1 + a_2 X_2 + \ldots + a_n X_n$$

- X_i = uncorrelated random variables, with unknown types of distribution, but with known mean values and standard deviations

- Then, the reliability index, β, can be calculated as follows,

$$\beta = \frac{a_0 + \sum_{i=1}^{n} a_i \mu_{X_i}}{\sqrt{\sum_{i=1}^{n} (a_i \sigma_{X_i})^2}}$$

Second Moment Reliability Index

$$\beta = \frac{a_0 + \sum_{i=1}^{n} a_i \mu_{X_i}}{\sqrt{\sum_{i=1}^{n} (a_i \sigma_{X_i})^2}}$$

The reliability index, β, depends on μ_{Xi} and σ_i only (it does not depend on the type of distribution). Therefore, this β is called a second moment measure of structural safety, because only the two first moments (mean and variance) are required. This formula is exact when all X_i are normal. Otherwise, it is only an approximation.

Reliability Index for a Non-linear Limit State Function

- Let's consider a non-linear limit state function

$$g\ (X_1, \ldots, X_n)$$

- X_i = uncorrelated random variables, with unknown types of distribution, but with known mean values and standard deviations

- Then, the limit state function can be linearized using a Taylor series expansion

$$g(X_1, X_2, \ldots, X_n) \approx g(x_1^*, \ldots, x_n^*) + \sum_{i=1}^{n} (X_i - x_i^*) \frac{\partial g}{\partial X_i}$$

where the derivatives are calculated at $(X_1^*, \ldots, X_n^*)$

First Order, Second Moment, Reliability Index

$$\beta = \frac{a_0 + \sum_{i=1}^{n} a_i \mu_{X_i}}{\sqrt{\sum_{i=1}^{n} (a_i \sigma_{X_i})^2}}$$

where $a_i = \frac{\partial g}{\partial X_i}$ calculated at $(X_1^*, \ldots, X_n^*)$.

But how to determine $(X_1^*, \ldots, X_n^*)$?

First Order, Second Moment, Mean Value Reliability Index

$$\beta = \frac{g\left(\mu_{x_1}, \mu_{x2}, \ldots\ldots \mu_{xn}\right)}{\sqrt{\sum_{i=1}^{n} \left(a_i \sigma_{x_i}\right)^2}}$$

where

$$a_i = \frac{\partial g}{\partial X_i}$$

The Taylor series expansion is calculated about the mean values.

Reliability Index

This reliability index is called a *first order, second moment, mean value, reliability index*.

- *first order* because we use first-order terms in the Taylor series expansion
- *second moment* because only means and variances are needed
- *mean value* because the Taylor series expansion is about the mean values

Example

- Consider the following beam

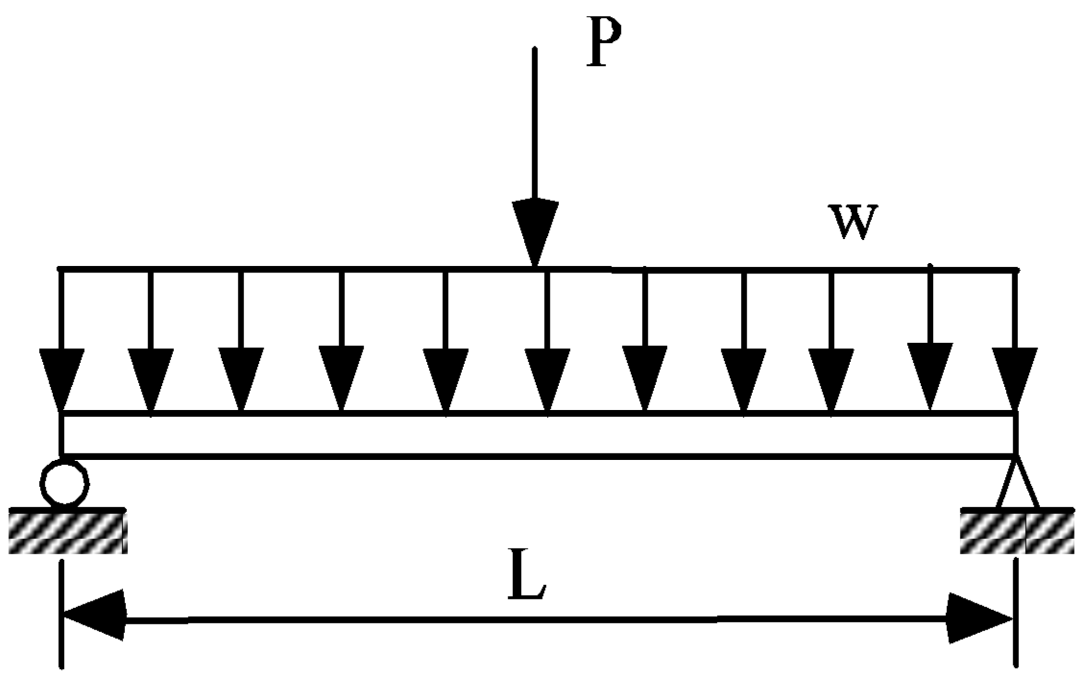


Figure 5-13 Beam considered in Example 5-1.

Example (continued)

Consider the simply-supported beam which is 12 ft long. The beam is subjected to uniformly distributed dead, live, and wind loads. The mean moment carrying capacity of the beam is 100 k-ft, and the coefficient of variation of the capacity is 13%. Calculate the probability of failure for the beam. Assume all random variables are normally distributed and uncorrelated

Example (continued)

For the loads

load	mean	σ
dead	0.95 k/ft	0.1 k/ft
live	1.5 k/ft	0.2 k/ft
wind	0.6 k/ft	0.12 k/ft

Example (continued)

The resistance in this case is the moment carrying capacity of the beam

$\mu_R = \mu_M = 100$ k-ft

$V_R = V_M = 0.13$

$\sigma_R = V_R \mu_R = (100)(0.13) = 13$ k-ft

Example (continued)

The limit state equation is

$$g = R - (M_D+M_L+M_W)$$

where M_D, M_L, and M_W represent the midspan moments caused by dead load, live load, and wind load, respectively.

Example (continued)

$$M_D = \frac{w_D L^2}{8} \qquad M_L = \frac{w_L L^2}{8} \qquad M_W = \frac{w_W L^2}{8} \qquad L = 12 \text{ feet}$$

Substituting these expressions into the limit state equation and substituting L = 12 feet, we get

$$g = R - 18(w_D + w_L + w_W)$$

Example (continued)

- Since the limit state equation is linear, and all variables are normally distributed and uncorrelated, to find β, we can use:

$$\beta = \frac{a_0 + \sum a_i \mu_{X_i}}{\sqrt{\sum (a_i \sigma_{X_i})^2}} = \frac{0 + 1(\mu_R) - 18(\mu_D + \mu_L + \mu_W)}{\sqrt{\sigma_R^2 + (-18\sigma_D)^2 + (-18\sigma_L)^2 + (-18\sigma_W)^2}}$$

$$= \frac{0 + 1(100) - 18(0.95 + 1.5 + 0.6)}{\sqrt{(13)^2 + \left((-18)(0.1)\right)^2 + \left((-18)(0.2)\right)^2 + \left((-18)(0.12)\right)^2}}$$

$$= 3.27$$

PF = Φ(-β) = Φ(-3.27) = 5.38x10-4

Example

Consider a reinforced concrete beam

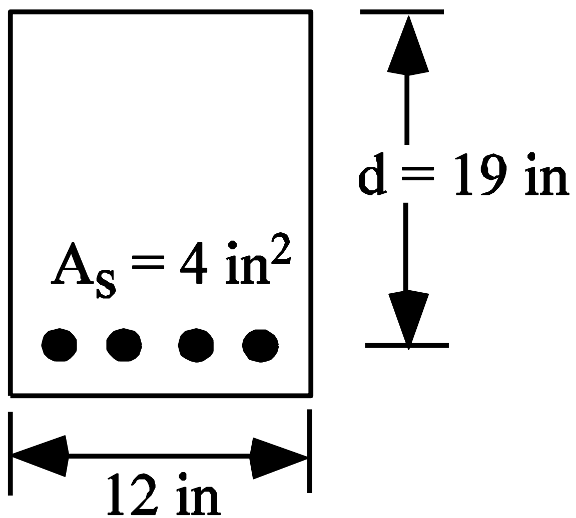


Figure 5-14 Cross-section of reinforced concrete beam considered in Example

Example (continued)

The moment-carrying capacity of the section is calculated using

$$M = A_s f_y \left(d - 0.59 \frac{A_s f_y}{f_c' b} \right) = A_s f_y d - 0.59 \frac{\left(A_s f_y\right)^2}{f_c' b}$$

Example (continued)

The limit state function is

$$g(A_s, f_y, f_c', Q) = A_s f_y d - 0.59 \frac{(A_s f_y)^2}{f_c' b} - Q$$

where Q is the moment (load effect) due to the applied load. The random variables in the problem are Q, f_y, f_c', and A_s.

Example (continued)

The distribution parameters and design parameters are given

	mean	nominal	λ	σ	V
fy	44 ksi	40 ksi	1.10	4.62 ksi	0.105
As	4.08 in^2	4 in^2	1.02	0.08 in^2	0.02
f_c'	3.12 ksi	3 ksi	1.04	0.44 ksi	0.14
Q	2052 k-in	2160 k-in	0.95	246 k-in	0.12

λ is the bias factor (ratio of mean value to nominal value). The values of d and b are assumed to be deterministic constants. Calculate the reliability index, β.

Example (continued)

For this problem, the limit state function is nonlinear. The Taylor expansion about the mean values yields the following linear function:

$$g(A_s, f_y, f_c', Q) \approx \left[\mu_{A_s}\mu_{f_y} d - 0.59\frac{\left(\mu_{A_s}\mu_{f_y}\right)^2}{\mu_{f_c'} b} - \mu_Q\right] + \left(A_s - \mu_{A_s}\right)\frac{\partial g}{\partial A_s}\bigg|_{\text{evaluated at mean values}}$$

$$+\left(f_y - \mu_{f_y}\right)\frac{\partial g}{\partial f_y}\bigg|_{\text{evaluated at mean values}} + \left(f_c' - \mu_{f_c'}\right)\frac{\partial g}{\partial f_c'}\bigg|_{\text{evaluated at mean values}}$$

$$+\left(Q - \mu_Q\right)\frac{\partial g}{\partial Q}\bigg|_{\text{evaluated at mean values}}$$

Example (continued)

To calculate β, the partial derivatives must be determined and the limit state function must be evaluated at the mean values of the random variables:

$$g(\mu_{A_s},\mu_{f_y},\mu_{f_c'},\mu_Q)=\mu_{A_s}\mu_{f_y}d-0.59\frac{\left(\mu_{A_s}\mu_{f_y}\right)^2}{\mu_{f_c'}b}-\mu_Q=851.0\ \ \text{k / in}$$

$$a_1=\left.\frac{\partial g}{\partial A_s}\right|_{\text{mean values}}=\left[f_y d-0.59\frac{\left(2A_s f_y^{\,2}\right)}{f_c' b}\right]\Bigg|_{\text{mean values}}=587.1\quad \text{k / in}$$

$$a_2=\left.\frac{\partial g}{\partial f_y}\right|_{\text{mean values}}=\left[A_s d-0.59\frac{\left(2f_y A_s^2\right)}{f_c' b}\right]\Bigg|_{\text{mean values}}=54.44\quad \text{in}^3$$

$$a_3=\left.\frac{\partial g}{\partial f_c'}\right|_{\text{mean values}}=\left[0.59\frac{\left(A_s f_y\right)^2}{(f_c')^2 b}\right]\Bigg|_{\text{mean values}}=162.8\quad \text{in}^3$$

$$a_4=\left.\frac{\partial g}{\partial Q}\right|_{\text{mean values}}=-1\Big|_{\text{mean values}}=-1$$

Example (continued)

So, substituting these results into the equation of β we get

$$\beta = \frac{g(\mu_{A_s}, \mu_{f_y}, \mu_{f_c'}, \mu_Q)}{\sqrt{\left((587.1)(\sigma_{A_s})\right)^2 + \left((54.44)(\sigma_{f_y})\right)^2 + \left((162.8)(\sigma_{f_c'})\right)^2 + \left((-1)(\sigma_Q)\right)^2}}$$

$$= \frac{851.0}{\sqrt{\left((587.1)(0.08)\right)^2 + \left((54.44)(4.62)\right)^2 + \left((162.8)(0.44)\right)^2 + \left((-1)(246)\right)^2}}$$

$$= \frac{851.0}{362.1} = 2.35$$

Comments on the First-Order, Second Moment Mean Value, Reliability Index

The mean value second moment method is based on approximating non-normal CDF's of the state variables by normal variables, for the simple case in which $g(R, Q) = R - Q$. The method has both advantages and disadvantages in the structural reliability analysis

- Easy to use.
- Does not require knowledge of the distributions of the random variables.

but

- Inaccurate results if the tails of the distribution functions cannot be approximated by a normal distribution.
- Invariance problem: the value of the reliability index depends on the specific form of the limit state function.

Comments on the First-Order, Second Moment Mean Value, Reliability Index

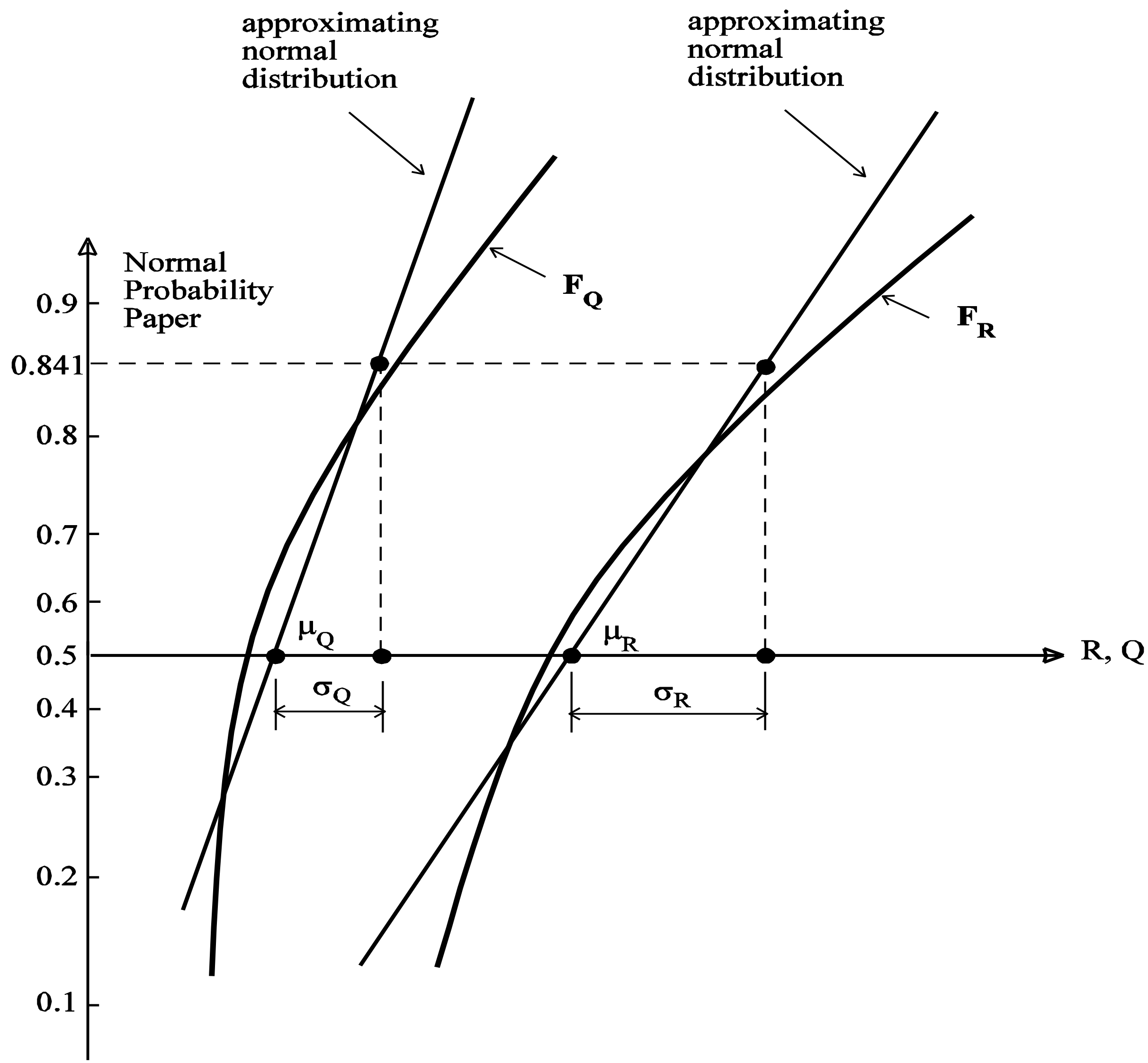


Figure 5-15 Mean value second moment formulation.

Comments on the First-Order, Second Moment Mean Value, Reliability Index

The calculation of the First Order, Second Moment, Mean Value Reliability Index depends on the **formulation of the problem**

Example

Consider the steel beam

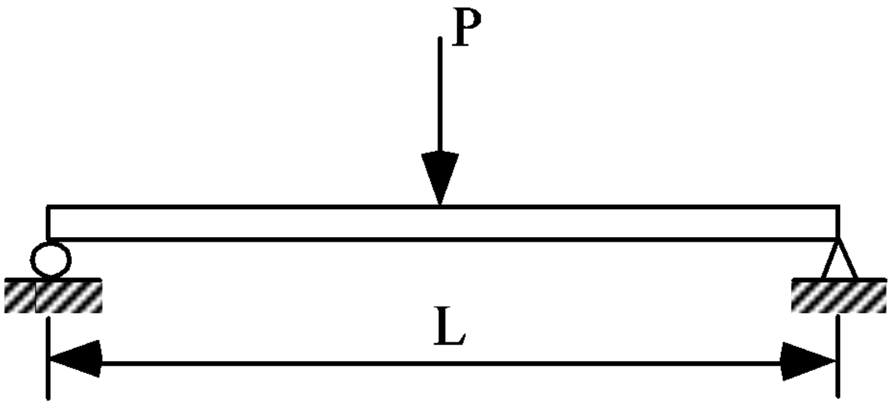


Figure 5-16 Steel beam considered in Example Problem 5-3.

Example (continued)

The steel beam is assumed to be compact with parameters Z (plastic modulus) and yield stress Fy. There are four random variables to consider: P, L, Z, Fy. It is assumed that the four variables are uncorrelated. The means and covariance matrix are given as

$$\{\mu_X\} = \begin{Bmatrix} \mu_P \\ \mu_L \\ \mu_Z \\ \mu_{F_y} \end{Bmatrix} = \begin{Bmatrix} 10 \quad \text{kN} \\ 8 \quad \text{m} \\ 100\text{x}10^{-6} \quad \text{m}^3 \\ 600\text{x}10^{3} \quad \text{kN}/\text{m}^2 \end{Bmatrix}$$

Example (continued)

$$[C_X] = \begin{bmatrix} 4\,kN^2 & 0 & 0 & 0 \\ 0 & 10x10^{-3}\ m^2 & 0 & 0 \\ 0 & 0 & 400x10^{-12}\ m^6 & 0 \\ 0 & 0 & 0 & 10x10^9\ (kN/m^2)^2 \end{bmatrix}$$

Example (continued)

To begin, consider a limit state function in terms of moments. We can write

$$g_1(Z, F_y, P, L) = ZF_y - \frac{PL}{4}$$

Example (continued)

Now, recall that the purpose of the limit state function is to define the boundary between the safe and unsafe domains, and the boundary corresponds to g = 0. So, if we divide g_1 by a positive quantity (for example, Z), then we are not changing the boundary or the regions in which the limit state function is positive or negative. So, an alternate limit state function (with units of stress) would be

$$g_2(Z, F_y, P, L) = F_y - \frac{PL}{4Z} = \frac{g_1(Z, F_y, P, L)}{Z}$$

Example (continued)

- Since both functions satisfy the requirements for a limit state function, they are both valid. Now let's calculate the reliability index for both functions
- For g_1, since it is nonlinear, the limit state function is linearized about the means. The results are

$$g_1 \approx \left[\mu_Z \mu_{F_y} - \frac{\mu_P \mu_L}{4}\right] + \mu_{F_y}(Z - \mu_Z) + \mu_Z(F_y - \mu_{F_y}) - \frac{\mu_L}{4}(P - \mu_P) - \frac{\mu_P}{4}(L - \mu_L)$$

$$\beta = 2.48$$

Example (continued)

For g_2, which is also nonlinear, again linearize about the mean values. The results are

$$g_2 \approx \left[\mu_{F_y} - \frac{\mu_P \mu_L}{4\mu_Z}\right] + \frac{\mu_P \mu_L}{4(\mu_Z)^2}(Z - \mu_Z) + (1)(F_y - \mu_{F_y}) - \frac{\mu_L}{4\mu_Z}(P - \mu_P) - \frac{\mu_P}{4\mu_Z}(L - \mu_L)$$

$$\beta = 3.48$$

Example - Conclusion

This example clearly demonstrates the "invariance" in the mean value second moment reliability index. In this example, the same fundamental limit state forms the basis for both limit state functions. Therefore, the probability of failure (as reflected by the reliability index) should be the same. It is possible to remove the invariance problem, and this is discussed in the next section.

Hasofer-Lind Reliability Procedure

Calculate β for a given limit state function $g(X_1, X_2, ..., X_n)$ where the random variables X_i are all *uncorrelated*. If the variables are correlated, then a transformation can be used to obtain uncorrelated variables. It is assumed that CDF’s are not available. For a linear limit state function, use Cornell’s formula.

$$\beta = \frac{a_0 + \sum_{i=1}^{n} a_i \mu_{X_i}}{\sqrt{\sum_{i=1}^{n} (a_i \sigma_{X_i})^2}}$$

Hasofer-Lind Reliability Index

Consider a limit state function $g(X_1, X_2, ...,X_n)$ where the random variables X_i

Each variable Xi has a corresponding reduced variable (or standard form) Zi, such as

$$Z_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}}$$

So Xi = mxi + Zi sxi

Hasofer-Lind Reliability Index

Replace X_i in the limit state function $g(X_1, X_2, ..., X_n)$ with

$$Xi = mxi + Zi\ sxi$$

So the new limit state function is

$$g\ (Z1, \ldots\ Zn) = 0$$

Find β as the shortest distance from (0,…0) to g(Z1, … Zn)=0 in the reduced variable space

Hasofer-Lind Reliability Index

- The big change comes if the limit state function is nonlinear.
- Then, iterations are required to find the design point $\{z_1^*, z_2^*, \ldots, z_n^*\}$ in reduced variable space such that β still corresponds to the shortest distance

Hasofer-Lind Reliability Index

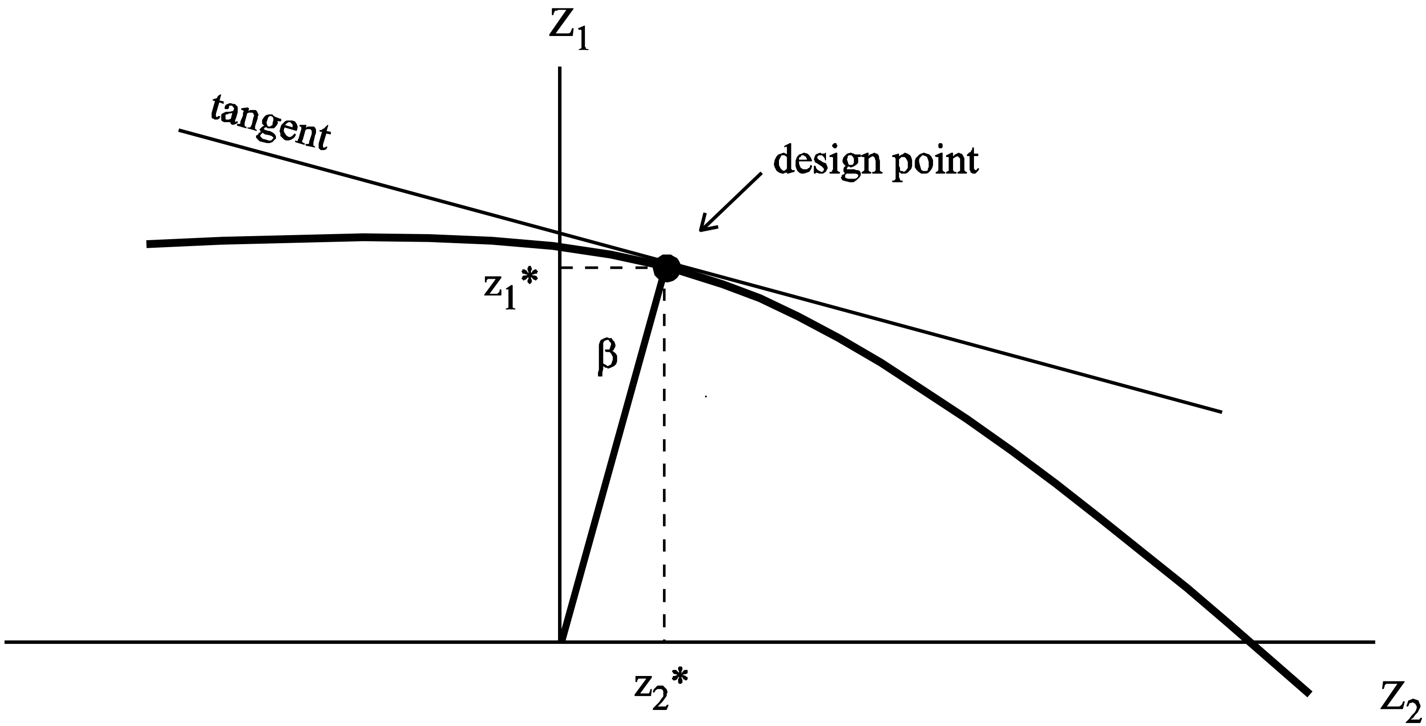


Figure 5-17 Hasofer-Lind reliability index.

Hasofer-Lind Reliability Index

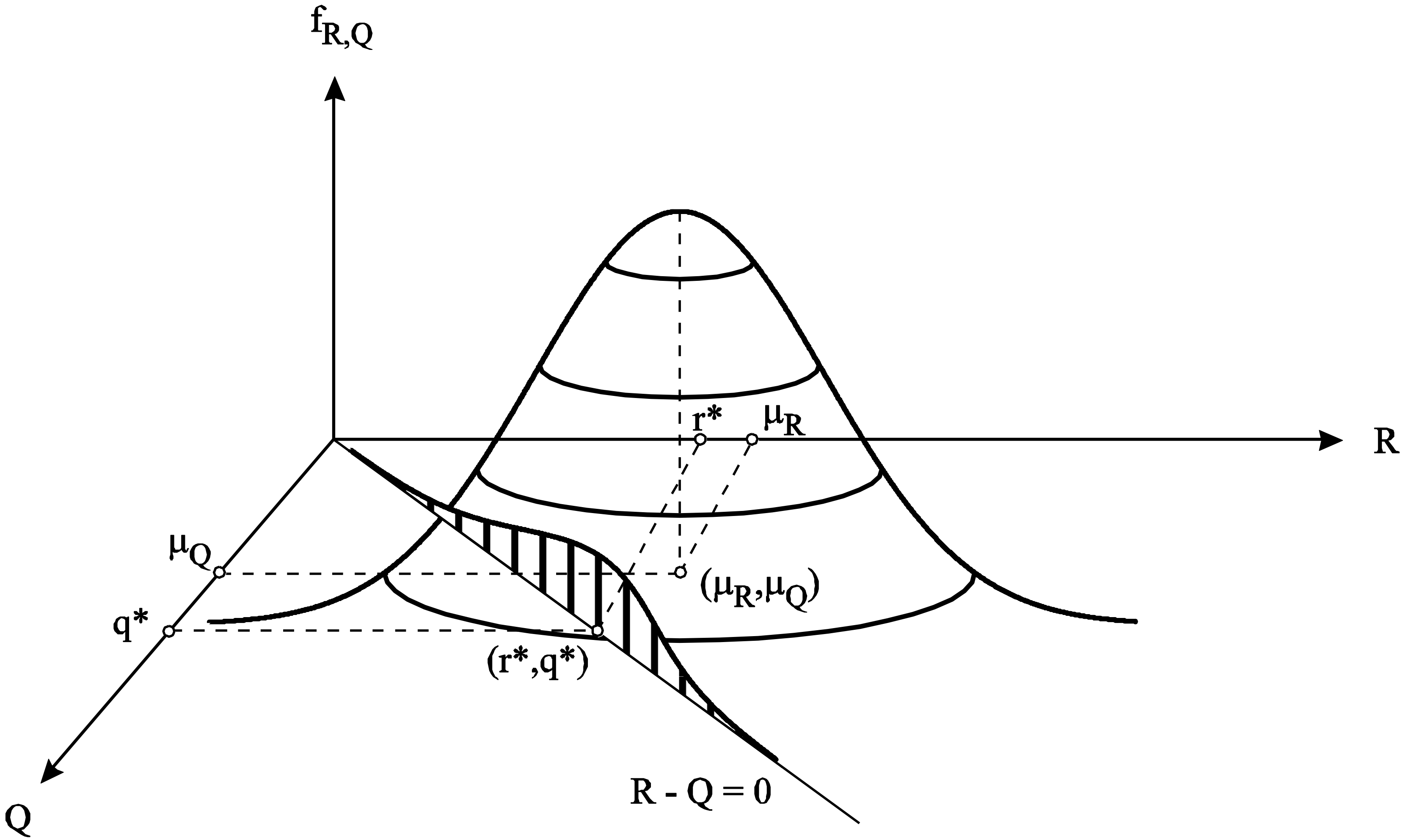


Figure 5-18 Design point on the failure boundary for the linear limit state function g=R-Q.

Hasofer-Lind Reliability Index

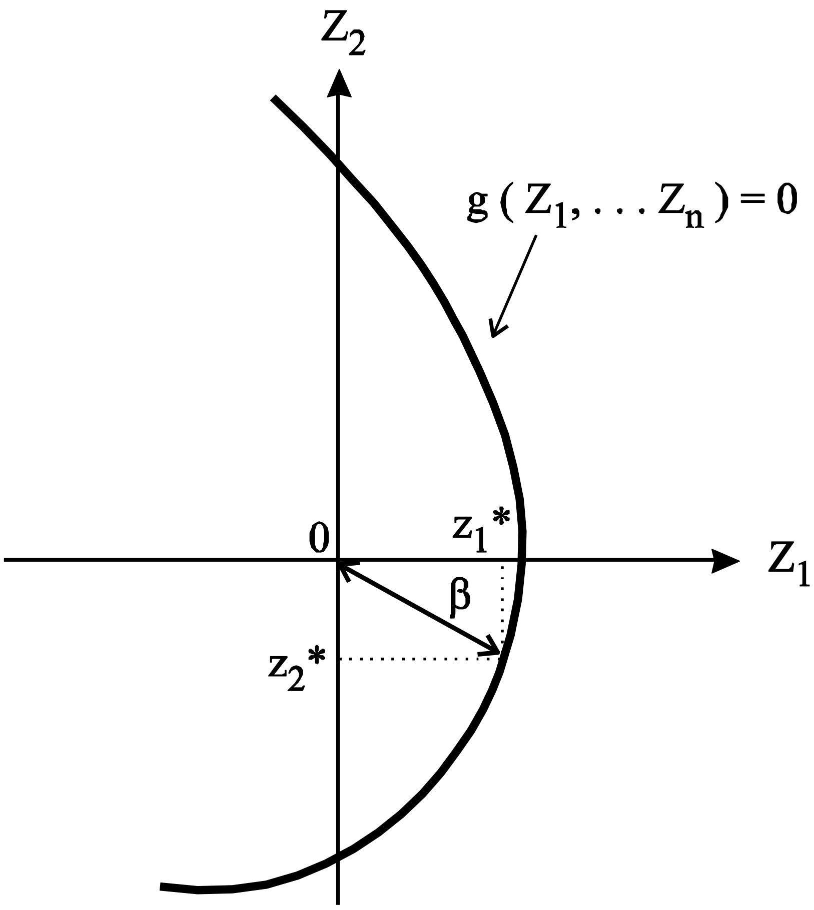


Figure 5-19 Design point and reliability index for a highly nonlinear limit state function.

Hasofer-Lind Reliability Index

The iterative procedure requires us to solve a set of (2n+1) simultaneous equations with (2n+1) unknowns:

- $\beta, \alpha_1, \alpha_2, \ldots, \alpha_n,$
- $z_1^*, z_2^*, \ldots, z_n^*$

where

$$\alpha_i = \frac{-\left.\frac{\partial g}{\partial Z_i}\right|_{\text{evaluated at design point}}}{\sqrt{\sum_{k=1}^{n}\left(\left.\frac{\partial g}{\partial Z_k}\right|_{\text{evaluated at design point}}\right)^2}}$$

$$\sum_{i=1}^{n}(\alpha_i)^2 = 1$$

$$z_i^* = \beta\alpha_i$$

$$\frac{\partial g}{\partial Z_i} = \frac{\partial g}{\partial X_i}\frac{\partial X_i}{\partial Z_i} = \frac{\partial g}{\partial X_i}\sigma_{X_i}$$

$$g(z_1^*, z_2^*, \ldots, z_n^*) = 0$$

Hasofer-Lind Reliability Index

There are two alternative procedures presented below for performing the iterative analysis. They will be identified as the "simultaneous equation" procedure and the "matrix" procedure. The steps in each procedure are summarized below.

Hasofer-Lind Reliability Index

- Simultaneous Equation Procedure:

1. Formulate the limit state function $g(X_1,\dots, X_n) = 0$ and appropriate parameters for all random variables involved.
2. Express the limit state function in terms of reduced variables Z_i, by replacing all X_i with $(\mu_i + Z_i \sigma_i)$
3. Express the limit state function in terms of β and α_i by replacing all Z_i with $\alpha_i \beta$
4. Express each α_i as a function of all α_i and β.

Hasofer-Lind Reliability Index

Calculate α_i

$$\alpha_i = \frac{-\left.\dfrac{\partial g}{\partial Z_i}\right|_{\text{evaluated at design point}}}{\sqrt{\displaystyle\sum_{k=1}^{n}\left(\left.\dfrac{\partial g}{\partial Z_k}\right|_{\text{evaluated at design point}}\right)^2}}$$

5. Initial cycle: Assume (guess) numerical values of β and all α_i. It is convenient to start with

$\alpha_1 = \ldots = \alpha_i = \ldots = \alpha_n$.

Hasofer-Lind Reliability Index

6. Solve the 2n+1 simultaneous equations for β and α_i.

$$g(z_1^*, z_2^*, \ldots, z_n^*) = 0$$

$$z_i^* = \beta\alpha_i \qquad \alpha_i = \frac{-\left.\dfrac{\partial g}{\partial Z_i}\right|_{\text{evaluated at design point}}}{\sqrt{\displaystyle\sum_{k=1}^{n}\left(\left.\dfrac{\partial g}{\partial Z_k}\right|_{\text{evaluated at design point}}\right)^2}}$$

7. Iterate until the β and α_i values converge.

Hasofer-Lind Reliability Index

- Matrix Procedure:
  1. Formulate the limit state function and appropriate parameters for all random variables X_i (i = 1,2,...,n) involved.
  2. Obtain an initial design point $\{x_i^*\}$ by assuming values for n-1 of the random variables X_i. (Mean values are often a reasonable initial choice.) Solve the limit state equation g = 0 for the remaining random variable. This ensures that the design point is on the failure boundary.
  3. Determine the reduced variables $\{z_i^*\}$ corresponding to the design point $\{x_i^*\}$ using

$$z_i^* = \frac{x_i^* - \mu_{X_i}}{\sigma_{X_i}}$$

Hasofer-Lind Reliability Index

4. Determine the partial derivatives of the limit state function with respect to the reduced variables. For convenience, define a column vector {G} as the vector whose elements are these partial derivatives multiplied by -1, i.e.,

$$\{G\} = \begin{Bmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{Bmatrix} \qquad G_i = -\left.\frac{\partial g}{\partial Z_i}\right|_{\text{evaluated at design point}}$$

5. Calculate an estimate of β using the following formula:

$$\beta = \frac{\{G\}^T\{z^*\}}{\sqrt{\{G\}^T\{G\}}} \qquad \{z^*\} = \begin{Bmatrix} z_1^* \\ z_2^* \\ \vdots \\ z_n^* \end{Bmatrix}$$

Hasofer-Lind Reliability Index

6. Calculate a column vector containing the sensitivity factors using

$$\{\alpha\} = \frac{\{G\}}{\sqrt{\{G\}^T\{G\}}}$$

7. Determine a new design point in reduced variables for n-1 of the variables using

$$z_i^* = \alpha_i \beta$$

8. Determine the corresponding design point values in original coordinates for the n-1 values in Step 7 using

$$x_i^* = \mu_{X_i} + z_i^* \sigma_{X_i}$$

Hasofer-Lind Reliability Index

9. Determine the value of the remaining random variable (i.e., the one not found in steps 7 and 8) by solving the limit state equation g = 0.
10. Repeat steps 3-9 until β and the design point $\{x_i^*\}$ converge.

Example

- Calculate the Hasofer-Lind reliability index for the 3-span continuous beam shown

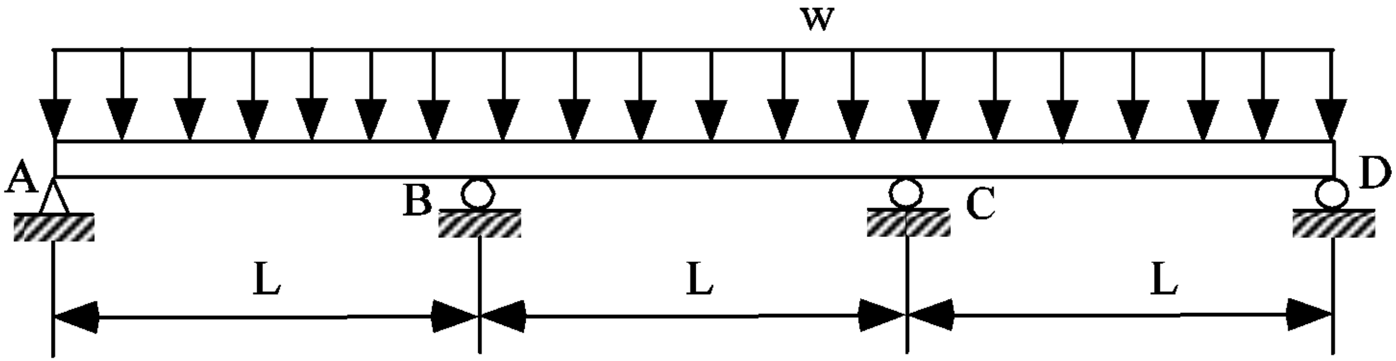


Figure 5-20 Beam considered in Example 5-4.

- To solve the problem, we will follow the steps in the simultaneous equation procedure

Example (continued)

- The random variables in the problem are:
- distributed load (w),
- span length (L),
- modulus of elasticity (E),
- moment of inertia (I).

The limit state to be considered is deflection, and the allowable deflection is specified as L/360. The maximum deflection is 0.0069 wL^4/EI, and it occurs at 0.446L from either end (AISC, 1986). The limit state function is

$$g(w, L, E, I) = \frac{L}{360} - 0.0069\frac{wL^4}{EI}$$

Example (continued)

Means and standard deviations of the random variables

variable	mean	standard deviation
w	10 kN/m	0.4 kN/m
L	5 m	~ 0
E	2×10^7 kN/m^2	0.5×10^7 kN/m^2
I	8×10^{-4} m^4	1.5×10^{-4} m^4

Example (continued)

- Express g as a function of reduced variables. First, substituting some numbers, g can be expressed as

$$g = 0 \quad \Rightarrow \quad \frac{5}{360} EI - 0.0069(5^4)w = 0 \quad \Rightarrow \quad EI - 310.5w = 0$$

- Define the reduced variables

$$Z_1 = \frac{I - \mu_I}{\sigma_I} \quad ; \quad Z_2 = \frac{E - \mu_E}{\sigma_E} \quad ; \quad Z_3 = \frac{w - \mu_w}{\sigma_w}$$

$$I = \mu_I + Z_1\sigma_I \quad ; \quad E = \mu_E + Z_2\sigma_E \quad ; \quad w = \mu_w + Z_3\sigma_w$$

Example (continued)

- Substitute into g

$$\left(\mu_E + Z_2\sigma_E\right)\left(\mu_I + Z_1\sigma_I\right) - 310.5\left(\mu_w + Z_3\sigma_w\right) = 0$$

$$\left(2x10^7 + Z_2(0.5x10^7)\right)\left(8x10^{-4} + Z_1(1.5x10^{-4})\right) - 310.5\left(10 + Z_3(0.4)\right) = 0$$

$$(3000)Z_1 + (4000)Z_2 + (750)Z_1Z_2 - (124.2)Z_3 + 12895 = 0$$

- Formulate g in terms of β and α

$$z_i^* = \beta\alpha_i$$

$$\Downarrow$$

$$3000\beta\alpha_1 + 4000\beta\alpha_2 + 750\beta^2\alpha_1\alpha_2 - 124.2\beta\alpha_3 + 12895 = 0$$

$$\beta = \frac{-12895}{3000\alpha_1 + 4000\alpha_2 + 750\beta\alpha_1\alpha_2 - 124.2\beta\alpha_3}$$

Example (continued)

- Calculate α_i values

$$\alpha_1 = \frac{-(3000 + 750\beta\alpha_2)}{\sqrt{(3000 + 750\beta\alpha_2)^2 + (4000 + 750\beta\alpha_1)^2 + (-124.2)^2}}$$

$$\alpha_2 = \frac{-(4000 + 750\beta\alpha_1)}{\sqrt{(3000 + 750\beta\alpha_2)^2 + (4000 + 750\beta\alpha_1)^2 + (-124.2)^2}}$$

$$\alpha_3 = \frac{-(-124.2)}{\sqrt{(3000 + 750\beta\alpha_2)^2 + (4000 + 750\beta\alpha_1)^2 + (-124.2)^2}}$$

Example (continued)

- The iterations start with a guess for β, α_1, α_2, α_3. For example, let's start with

$$\alpha_1 = \alpha_2 = -\sqrt{0.333} = -0.58 \quad ; \quad \alpha_3 = \sqrt{0.333} = 0.58$$

and let $\beta = 3$

Example (continued)

- The iterations are summarized. Notice that between iterations 5 and 6, the values change very little, so the solution has converged. Faster convergence occurs when the correct signs for the α_i's are used (+ for load, - for resistance)

	Initial Guess	Iteration #					
		1	2	3	4	5	6
β	3	3.664	3.429	3.213	3.175	3.173	3.173
α_1	-0.58	-0.532	-0.257	-0.153	-0.168	-0.179	-0.182
α_2	-0.58	-0.846	-0.965	-0.988	-0.985	-0.983	-0.983
α_3	+0.58	0.039	0.047	0.037	0.034	0.034	0.034

So, the calculated reliability index is approximately 3.17

Reliability Analysis Methods

- Linear limit state function
  - All normal random variables – use Cornell's formula – result is exact
  - Some are non-normal random variables - use Cornell's formula – result is approximated
  - Some are non-normal random variables – use Rackwitz-Fiessler procedure – results are close to exact
- Non-linear limit state function
  - Use Hasofer-Lind procedure
- Any type
  - Use Monte Carlo simulations – accuracy depends on number of runs
  - Use Rosenblueth 2n+1 method – approximate method, recommended when each run takes a long computer time

RACKWITZ-FIESSLER PROCEDURE

A procedure to calculate a reliability index when some of the random variable are not normal. The cumulative distribution functions must be known for all the variables involved.

The basic idea:

Replace each non-normal random variable with a normal variable such that CDF and PDF are the same for replaced and replacing variables at the so called "design point". Coordinates of the design point are found in iterations.

RACKWITZ-FIESSLER PROCEDURE - Steps

1. Guess coordinates of the design point (you can start with mean values)
2. Replace non-normal variables with normal such their CDF and PDF are the same at the design point.
3. Calculate β using Cornell's formula (for a linear limit state function)
4. Calculate coordinates of the new design point.
5. Go to step 2. Stop when the required accuracy is reached.

Mean and σ of approximating normal distribution

Suppose that a particular random variable X with mean μ_X and standard deviation σ_X is described by a cumulative distribution function $F_X(x)$ and a probability density function $f_X(x)$

To obtain the "equivalent normal" mean μ_X^e and standard deviation σ_X^e, we require that the CDF and PDF of the actual function be equal to the normal CDF and normal PDF at the value of the variable x* on the failure boundary described by g = 0.

Mathematically, these requirements are expressed as

$$F_X(x^*) = \Phi\left(\frac{x^* - \mu_X^e}{\sigma_X^e}\right) \qquad f_X(x^*) = \frac{1}{\sigma_X^e}\phi\left(\frac{x^* - \mu_X^e}{\sigma_X^e}\right)$$

RACKWITZ-FIESSLER PROCEDURE

- Suppose that a particular random variable X with mean μ_X and standard deviation σ_X is described by a cumulative distribution function $F_X(x)$ and a probability density function $f_X(x)$

- To obtain the "equivalent normal" mean μ_X^e and standard deviation σ_X^e, we require that the CDF and PDF of the actual function be equal to the normal CDF and normal PDF at the value of the variable x* on the failure boundary described by g = 0.

- Mathematically, these requirements are expressed as

$$F_X(x^*) = \Phi\left(\frac{x^* - \mu_X^e}{\sigma_X^e}\right) \qquad f_X(x^*) = \frac{1}{\sigma_X^e}\phi\left(\frac{x^* - \mu_X^e}{\sigma_X^e}\right)$$

RACKWITZ-FIESSLER PROCEDURE

By manipulating these equations :

$$\mu_X^e = x^* - \sigma_X^e \left[\Phi^{-1}\left(F_X(x^*) \right) \right]$$

$$\sigma_X^e = \frac{1}{f_X(x^*)} \phi\left(\frac{x^* - \mu_X^e}{\sigma_X^e} \right) = \frac{1}{f_X(x^*)} \phi\left[\Phi^{-1}\left(F_X(x^*) \right) \right]$$

RACKWITZ-FIESSLER PROCEDURE

Two Uncorrelated Variables

1. Let the limit state function be $g = R - Q$. The design point (R^*, Q^*) on the safety boundary, so $R^* = Q^*$.

2. Guess an initial value of R^*, between μ_R and μ_Q.
   Then $Q^* = R^*$.

3. Approximate F_R and F_Q by normal F_R^e and F_Q^e such that,

$$f_R^e(R^*) = f_R(R^*)$$

$$F_R^e(R^*) = F_R(R^*)$$

$$f_Q^e(Q^*) = f_Q(Q^*)$$

$$F_Q^e(Q^*) = F_Q(Q^*)$$

RACKWITZ-FIESSLER PROCEDURE

4. Calculate the mean and σ of the approximating normal distributions

$$\sigma_R^{\,e} = \frac{\phi\left\{\Phi^{-1}\left[F_R\left(R^*\right)\right]\right\}}{f_R\left(R^*\right)}$$

$$\mu_R^{\,e} = R^* - \sigma_R^{\,e}\,\Phi^{-1}\left[F_R\left(R^*\right)\right]$$

$$\sigma_Q^{\,e} = \frac{\phi\left\{\Phi^{-1}\left[F_Q\left(Q^*\right)\right]\right\}}{f_Q\left(Q^*\right)}$$

$$\mu_Q^{\,e} = Q^* - \sigma_Q^{\,e}\,\Phi^{-1}\left[F_Q\left(Q^*\right)\right]$$

RACKWITZ-FIESSLER PROCEDURE

5. Calculate β for the approximating normal distributions

$$\beta = \frac{\mu_R^{\,e} - \mu_Q^{\,e}}{\sqrt{(\sigma_R^{\,e})^2 + (\sigma_Q^{\,e})^2}}$$

6. Calculate the new design point

$$R^* = \mu_R^{\,e} - \frac{\beta(\sigma_R^{\,e})^2}{\sqrt{(\sigma_R^{\,e})^2 + (\sigma_Q^{\,e})^2}}$$

$$Q^* = \mu_Q^{\,e} + \frac{\beta(\sigma_Q^{\,e})^2}{\sqrt{(\sigma_R^{\,e})^2 + (\sigma_Q^{\,e})^2}}$$

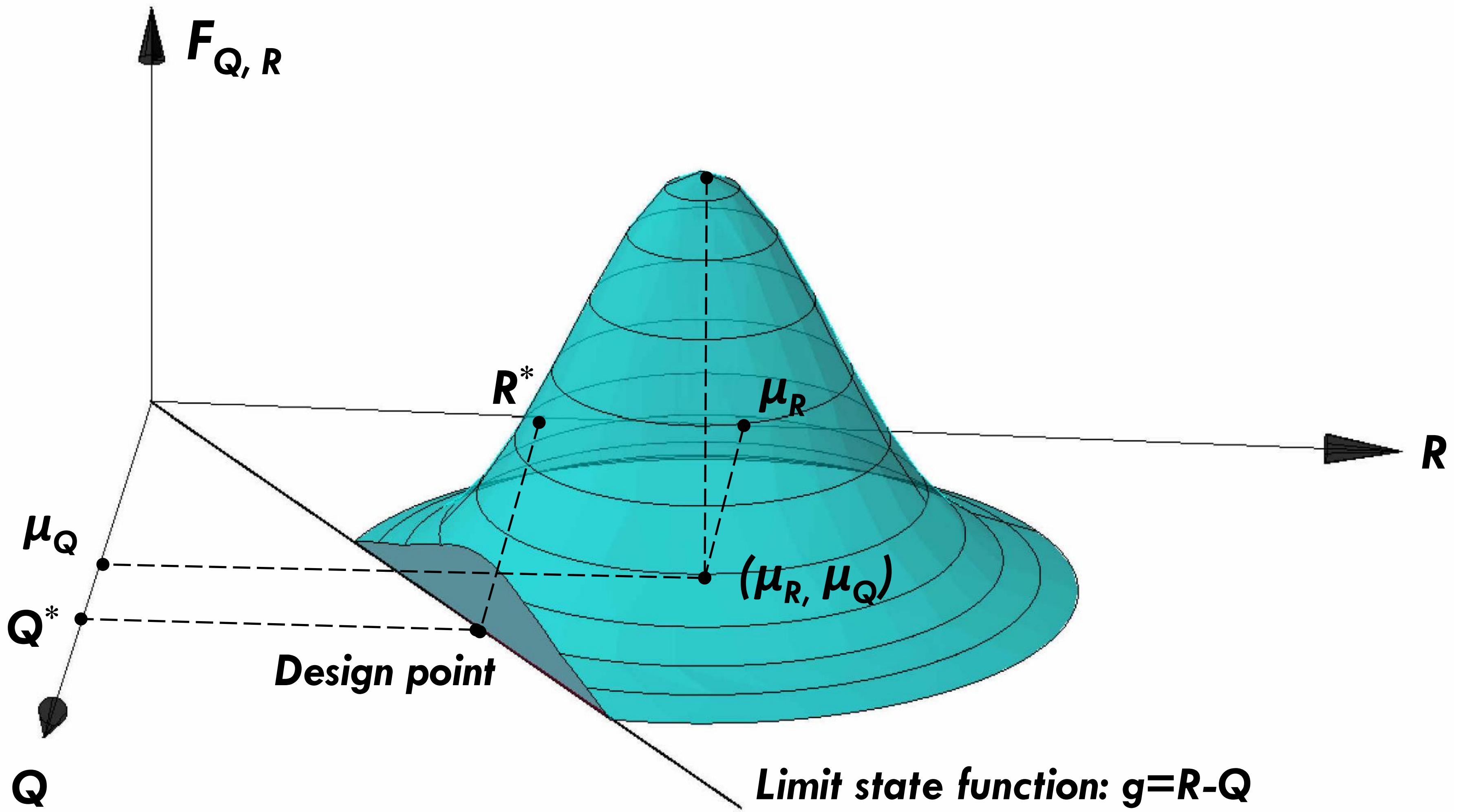
7. Go to step 2, and repeat the calculations until $\beta_{new} = \beta_{old}$

How to Determine Load and Resistance Factors?

- If failure is to occur – what are the most likely values of load and resistance?
- These values are the coordinates of the so called “design point”

Fundamental case

- Space of State Variables



Design Point - Load

$$Q^* = \mu_Q + \beta \frac{\sigma_Q^2}{\sqrt{\sigma_R^2 + \sigma_Q^2}}$$

μ_Q = Mean value of load

σ_R = Standard deviation of resistance

σ_Q = Standard deviation of load

β = Reliability Index

Design Point - Resistance

$$R^* = \mu_R - \beta \frac{\sigma_R^2}{\sqrt{\sigma_R^2 + \sigma_Q^2}}$$

μ_R = Mean value of resistance

σ_R = Standard deviation of resistance

σ_Q = Standard deviation of load

β = Reliability Index

Load Factor, γ_Q

$$\gamma_Q = \frac{Q^*}{Q_n}$$

Q_n = nominal value of Q

Q^* = coordinate of the design point for Q

Resistance Factors, φ

$$\varphi = \frac{R^*}{R_n}$$

R_n = nominal value of R

R^* = coordinate of the design point for R

RACKWITZ-FIESSLER PROCEDURE

- <u>Matrix Procedure :</u>

1. Formulate the limit state function. Determine the probability distributions and appropriate parameters for all random variables X_i
2. Obtain an initial design point $\{x_i^*\}$ by assuming values for <u>n-1</u> of the random variables X_i. (Mean values are often a reasonable choice.) Solve the limit state equation g = 0 for the remaining random variable. This ensures that the design point is on the failure boundary.
3. For each of the design point values x_i^* corresponding to a non-normal distribution, determine the equivalent normal mean and standard deviation. If one or more x_i^* values correspond to a normal distribution, then the equivalent normal parameters are simply the actual parameters.

RACKWITZ-FIESSLER PROCEDURE

4. Determine the reduced variables $\{z_i{*}\}$ corresponding to the design point $\{x_i{*}\}$ using

$$z_i^* = \frac{x_i^* - \mu_{X_i}^e}{\sigma_{X_i}^e}$$

5. Determine the partial derivatives of the limit state function with respect to the reduced variables.

$$\{G\} = \begin{Bmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{Bmatrix} \qquad G_i = -\left.\frac{\partial g}{\partial Z_i}\right|_{\text{evaluated at design point}}$$

RACKWITZ-FIESSLER PROCEDURE

6. Calculate an estimate of β using the following formula

$$\beta = \frac{\{G\}^T\{z^*\}}{\sqrt{\{G\}^T\{G\}}}$$

$$\{z^*\} = \begin{Bmatrix} z_1^* \\ z_2^* \\ \vdots \\ z_n^* \end{Bmatrix}$$

For a <u>linear</u> limit state function

$$\beta = \frac{a_0 + \sum_{i=1}^{n} a_i \mu_{X_i}^e}{\sqrt{\sum_{i=1}^{n} (a_i \sigma_{X_i}^e)^2}}$$

RACKWITZ-FIESSLER PROCEDURE

7. Calculate a column vector containing the sensitivity factors using

$$\{\alpha\} = \frac{\{G\}}{\sqrt{\{G\}^T \{G\}}}$$

8. Determine a new design point in reduced variables for n-1 of the variables using

$$z_i^* = \alpha_i \beta$$

9. Determine the corresponding design point values in original coordinates for the n-1 values in Step 7 using

$$x_i^* = \mu_{X_i}^e + z_i^* \sigma_{X_i}^e$$

RACKWITZ-FIESSLER PROCEDURE

10. Determine the value of the remaining random variable (i.e., the one not found in steps 8 and 9) by solving the limit state function $g = 0$.

11. Repeat steps 3-10 until β and the design point $\{x_i^*\}$ converge.

Example

- The modified matrix procedure is demonstrated on a simple case of two uncorrelated variables. Let R be the resistance and Q be the load effect. The limit state function is

$$g(R, Q) = R - Q$$

R lognormally distributed $\mu_R = 200$ and $\sigma_R = 20$.

Q extreme Type I distribution $\mu_Q = 100$ and $\sigma_Q = 12$.

Objective = calculate β

Example (continued)

1. Formulate limit state function and cumulative distribution functions.

2. Initial design point : Try r* = 150 - arbitrary guess

   From the limit state equation g = 0, q* = 150

3. Determine equivalent normal parameters

$$\sigma^2_{\ln R} = \ln\left(1 + \frac{\sigma^2_R}{\mu^2_R}\right) = 9.95\text{x}10^{-3} \quad \Rightarrow \quad \sigma_{\ln R} = 0.0998$$

$$\mu_{\ln R} = \ln(\mu_R) - 0.5\sigma^2_{\ln R} = 5.29$$

$$\mu^e_R = r^*\left[1 - \ln(r^*) + \mu_{\ln R}\right]$$

$$\sigma^e_R = r^*\sigma_{\ln R} = (150)(0.0998) = 15.0$$

$$= (150)\left[1 - \ln(150) + 5.29\right]$$

$$= 192$$

Example (continued)

Q : extreme type I distribution

$$F_Q(q) = \exp\left[-\exp\left(-a(q-u)\right)\right]$$

$$f_Q(q) = a\left\{\exp\left(-a(q-u)\right)\right\}\exp\left[-\exp\left(-a(q-u)\right)\right]$$

a and u are distribution parameters related to the mean and standard deviation of Q

$$u = \mu_Q - \frac{0.5772}{a} \quad ; \quad a = \sqrt{\frac{\pi^2}{6\sigma_Q^2}}$$

Plugging in the values of μ_Q and σ_Q, we find a=0.107 and u=94.6.

$$F_Q(q^*) = 0.997 \quad ; \quad f_Q(q^*) = 2.86x10^{-4}$$

Example (continued)

$$\sigma_Q^e = \frac{1}{f_Q(q^*)}\phi\left[\Phi^{-1}\left(F_Q(q^*)\right)\right] = \frac{1}{2.86\text{x}10^{-4}}\phi\left[\Phi^{-1}\left(0.997\right)\right] = 28.9$$

$$\mu_Q^e = q^* - \sigma_Q^e\left[\Phi^{-1}\left(F_Q(q^*)\right)\right] = 69.5$$

4. Determine the values of the reduced variables

z_1* : reduced variable for r*

z_2* : reduced variable for q*

$$z_1^* = \frac{r^* - \mu_R^e}{\sigma_R^e} = -2.83 \quad ; \quad z_2^* = \frac{q^* - \mu_Q^e}{\sigma_Q^e} = 2.78$$

Example (continued)

5. Determine the {G} vector

$$G_1 = -\left.\frac{\partial g}{\partial Z_1}\right|_{\{z_i^*\}} = -\left.\frac{\partial g}{\partial R}\right|_{\{x_i^*\}} \sigma_R^e = -1\sigma_R^e$$

$$G_2 = -\left.\frac{\partial g}{\partial Z_2}\right|_{\{z_i^*\}} = -\left.\frac{\partial g}{\partial Q}\right|_{\{x_i^*\}} \sigma_Q^e = +1\sigma_Q^e$$

6. Calculate an estimate of β.

$$\beta = \frac{\{G\}^T\{z^*\}}{\sqrt{\{G\}^T\{G\}}} = 3.78$$

Example (continued)

7. Calculate $\{\alpha\}$:

$$\{\alpha\} = \frac{\{G\}}{\sqrt{\{G\}^T\{G\}}} = \begin{Bmatrix} -0.460 \\ 0.888 \end{Bmatrix}$$

8. Determine new values of z_i* for n-1 of the variables. For example

$$z_1^* = \alpha_1 \beta = (-0.460)(3.78) = -1.74$$

9. Determine r* using the updated z_1*:

$$r^* = \mu_R^e + z_1^* \sigma_R^e = 166$$

10. Determine the value of q* using the limit state equation g = 0. For this case, q* = r* = 166

Example (continued)

11. Iterate until the value of β and the design point converge

	Iteration #		
	1	2	3
r^*	150	166	168
q^*	150	166	168
↓			
β	3.78	3.76	3.76
↓			
r^*	166	168	168
q^*	166	168	168

RACKWITZ-FIESSLER PROCEDURE GRAPHICAL PROCEDURE

- Can be applied when the CDFs of the basic variables are available as plots on normal probability paper.
- Each non-normal variable is approximated by a normal distribution, which is represented by a straight line.
- The value of the CDF of the approximating normal variable is the same at the design point as that of the original distribution.
- On normal probability paper this means that the straight line intersects with the original CDF at the design point.
- Since the PDF is a tangent (first derivative) of the CDF, the straight line (approximating normal) is tangent to the original CDF at the design point.
- The parameters of the approximating normal distribution (mean and standard deviation) can be read directly from the graph.

RACKWITZ-FIESSLER PROCEDURE GRAPHICAL PROCEDURE

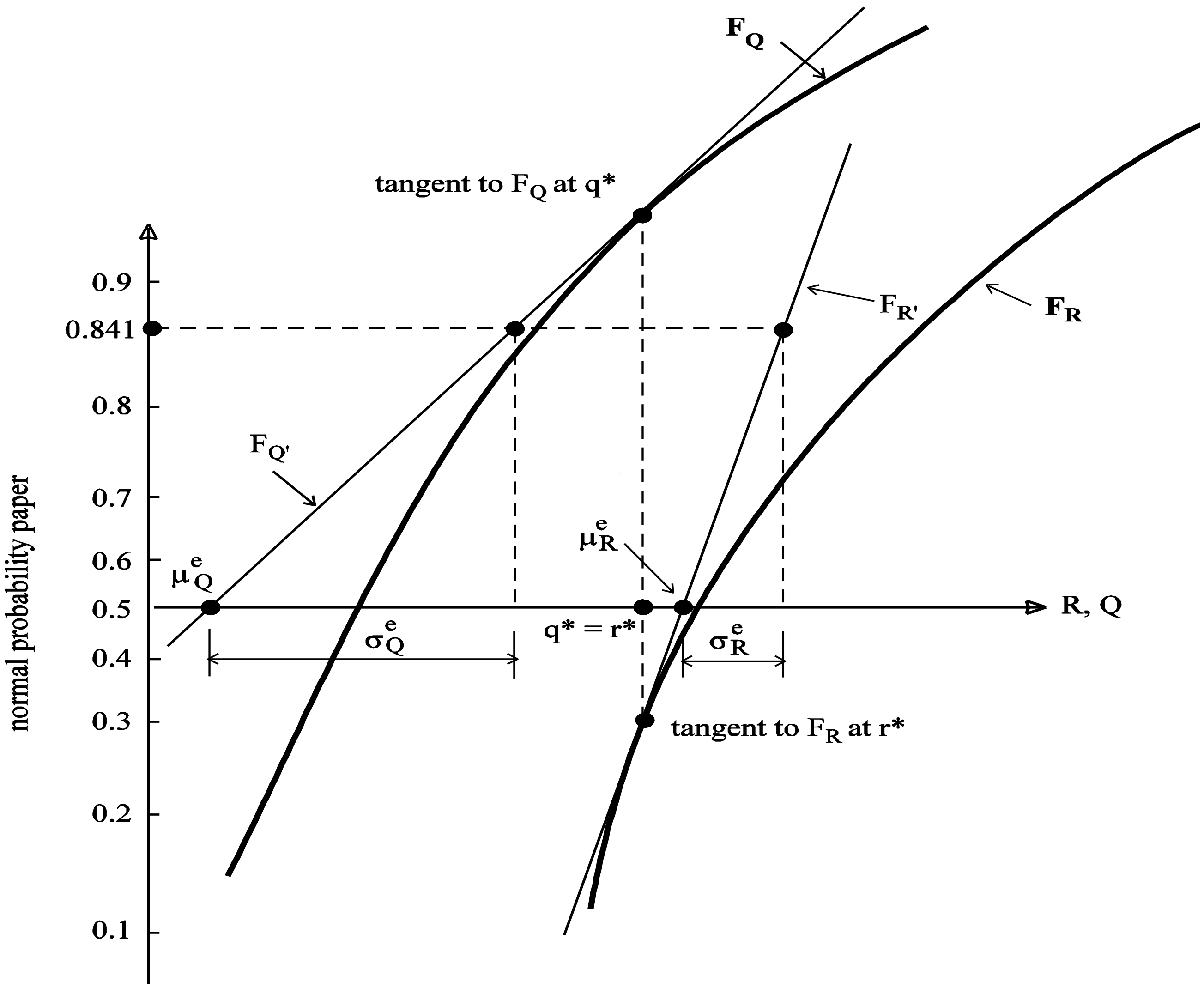
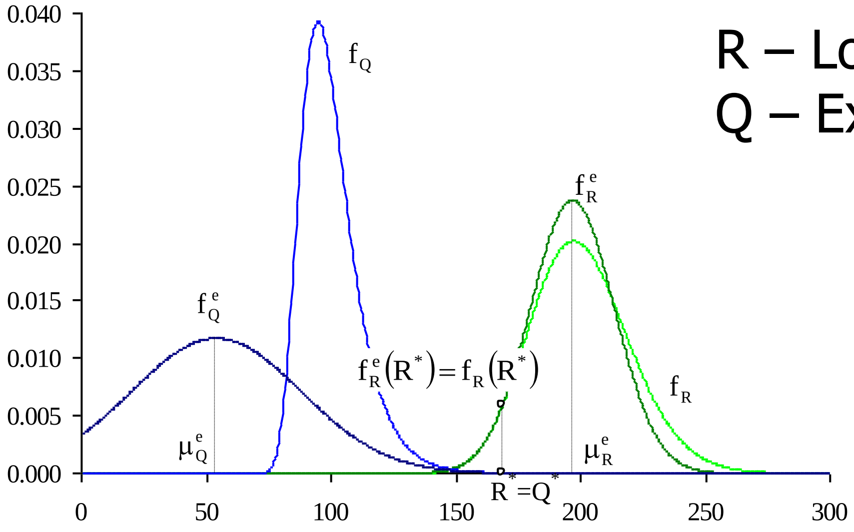


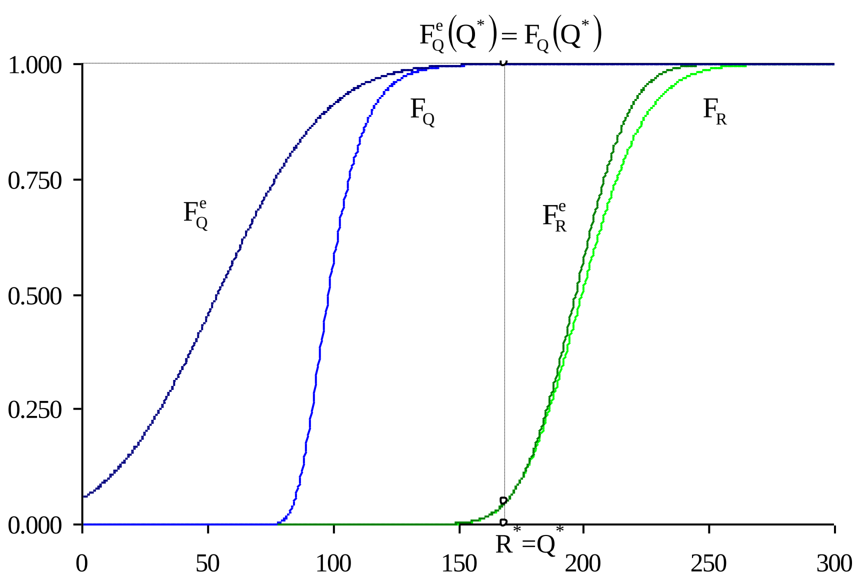
Figure 5-22 Graphical illustration of Rackwitz-Fiesslerr procedure.

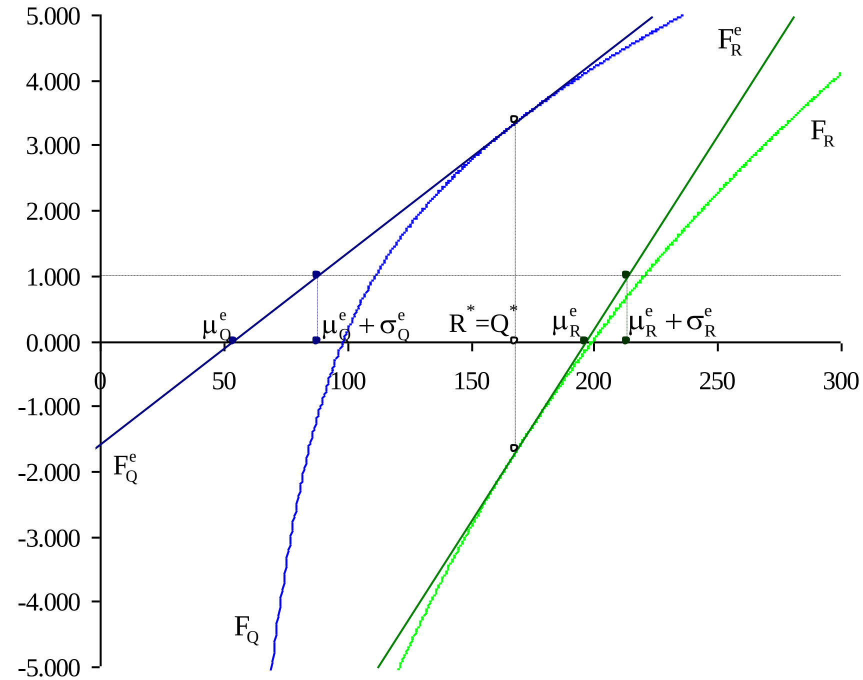
Example of Graphical Procedure

R – Lognormal

Q – Extreme Type I







EXAMPLE

We will demonstrate the calculation of the reliability index using the graphical procedure for the limit state function

$$g(R, Q) = R - Q$$

where

R = variable representing resistance

Q = variable representing the load effect

The CDF's for R and Q are plotted on normal probability paper

EXAMPLE continued

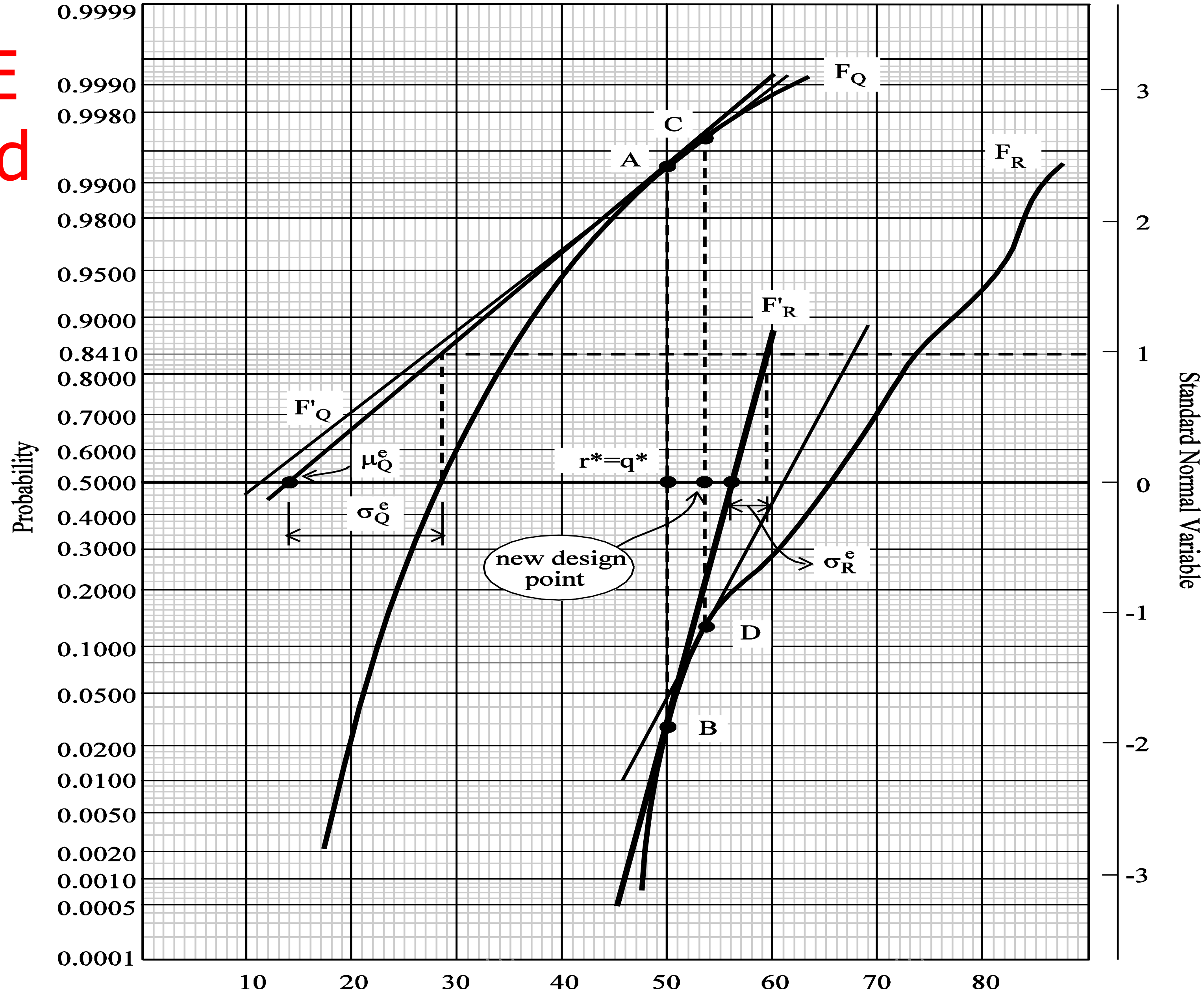


Figure 5-23 Graphical solution for Example Problem 5-12.

EXAMPLE (continued)

Basics steps for the graphical procedure

1. Guess the initial value of the design point, e.g. assume r* = q* = 50 ksi. Mark points A and B on the plots of F_R and F_Q, respectively
2. Plot tangents to F_R and F_Q at A and B
3. Read directly from the graph

$$\mu_R^e = 56 \text{ ksi} \qquad \sigma_R^e = 3.5 \text{ ksi}$$

$$\mu_Q^e = 14 \text{ ksi} \qquad \sigma_Q^e = 14.5 \text{ ksi}$$

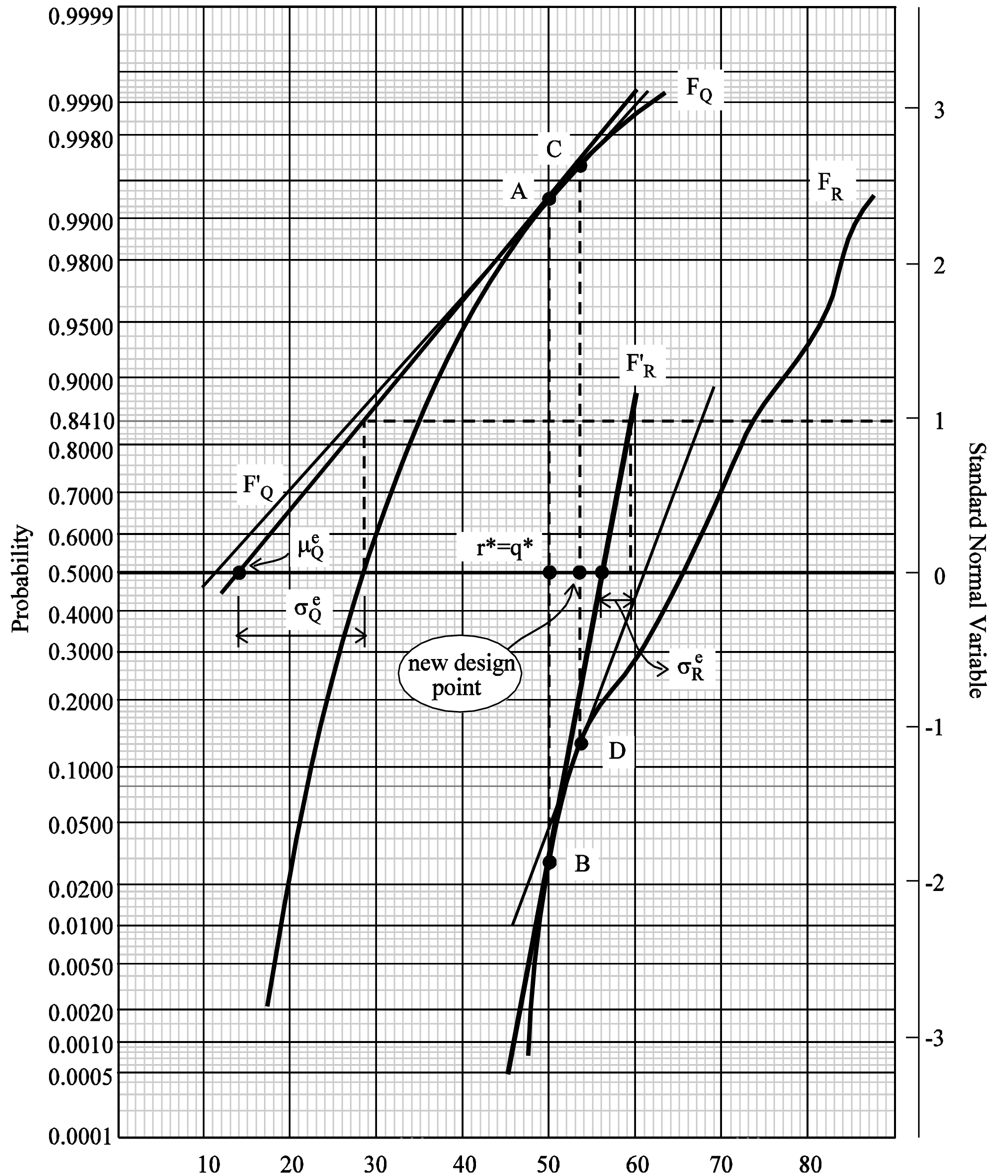


Figure 5-23 Graphical solution for Example Problem 5-12.

EXAMPLE (continued)

4. Calculate β

$$\beta = \frac{\mu_R^e - \mu_Q^e}{\sqrt{\left(\sigma_R^e\right)^2 + \left(\sigma_Q^e\right)^2}} = \frac{56 - 14}{\sqrt{(3.5)^2 + (14.5)^2}} = 2.82$$

5. Calculate new design point.

The value of q* is found from the requirement g = 0. Therefore, q* = r*.

$$r^* = \mu_R^e - \frac{\left(\sigma_R^e\right)^2 \beta}{\sqrt{(\sigma_R^e)^2 + (\sigma_Q^e)^2}} = 56 - \frac{(3.5)^2(2.82)}{\sqrt{(3.5)^2 + (14.5)^2}} = 53.7 \quad \text{ksi}$$

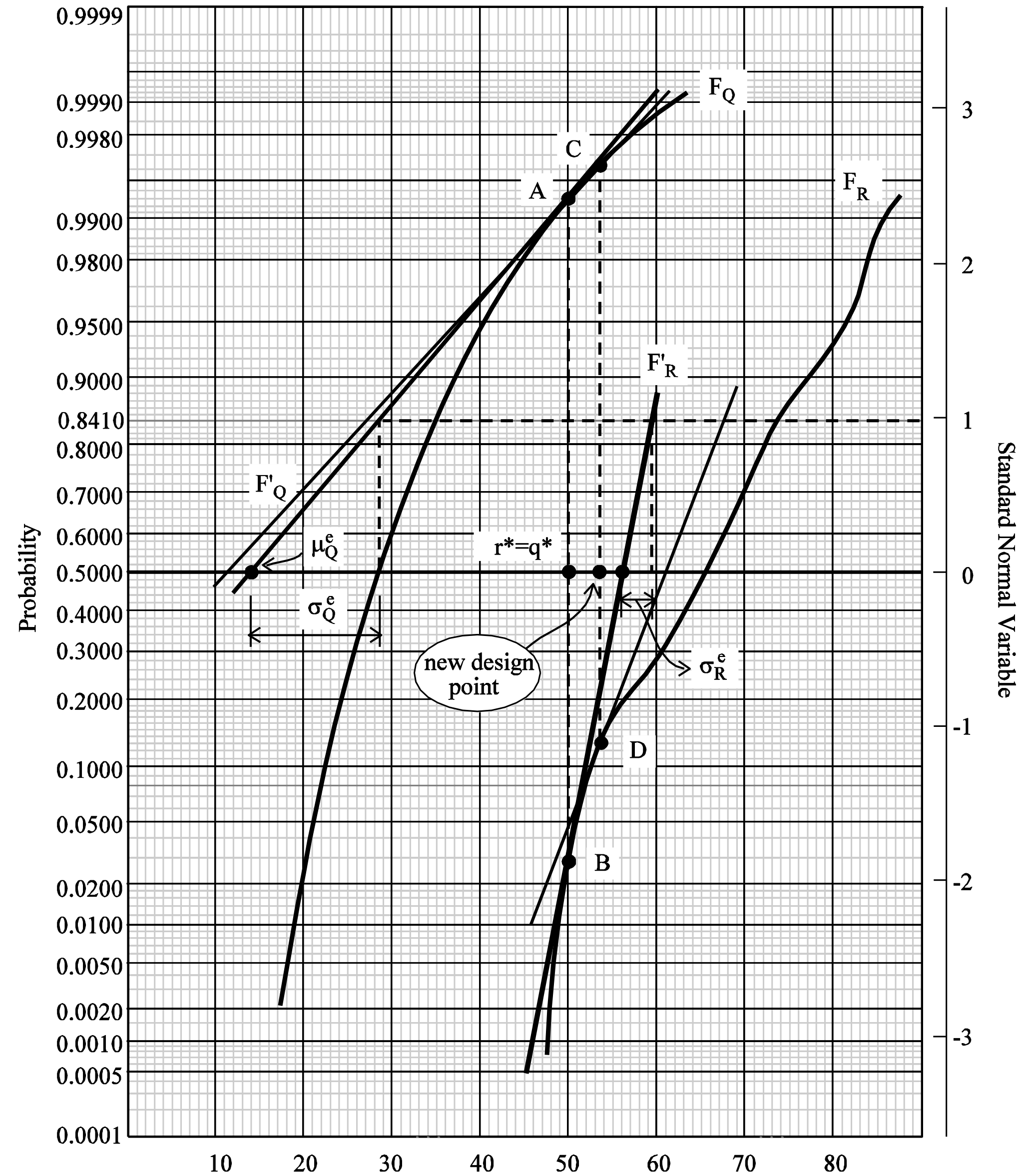


Figure 5-23 Graphical solution for Example Problem 5-12.

EXAMPLE (continued)

6. Plot tangents to F_R and F_Q at C and D using the updated design point.

7. Read directly from the graph

$$\mu_R^e = 61 \text{ ksi} \qquad \sigma_R^e = 6.5 \text{ ksi}$$

$$\mu_Q^e = 11.5 \text{ ksi} \qquad \sigma_Q^e = 15.5 \text{ ksi}$$

8. Calculate new β and design point,

$$\beta = 2.94 \qquad r^* = q^* = 53.6$$

9. The process would continue until β converges to a value

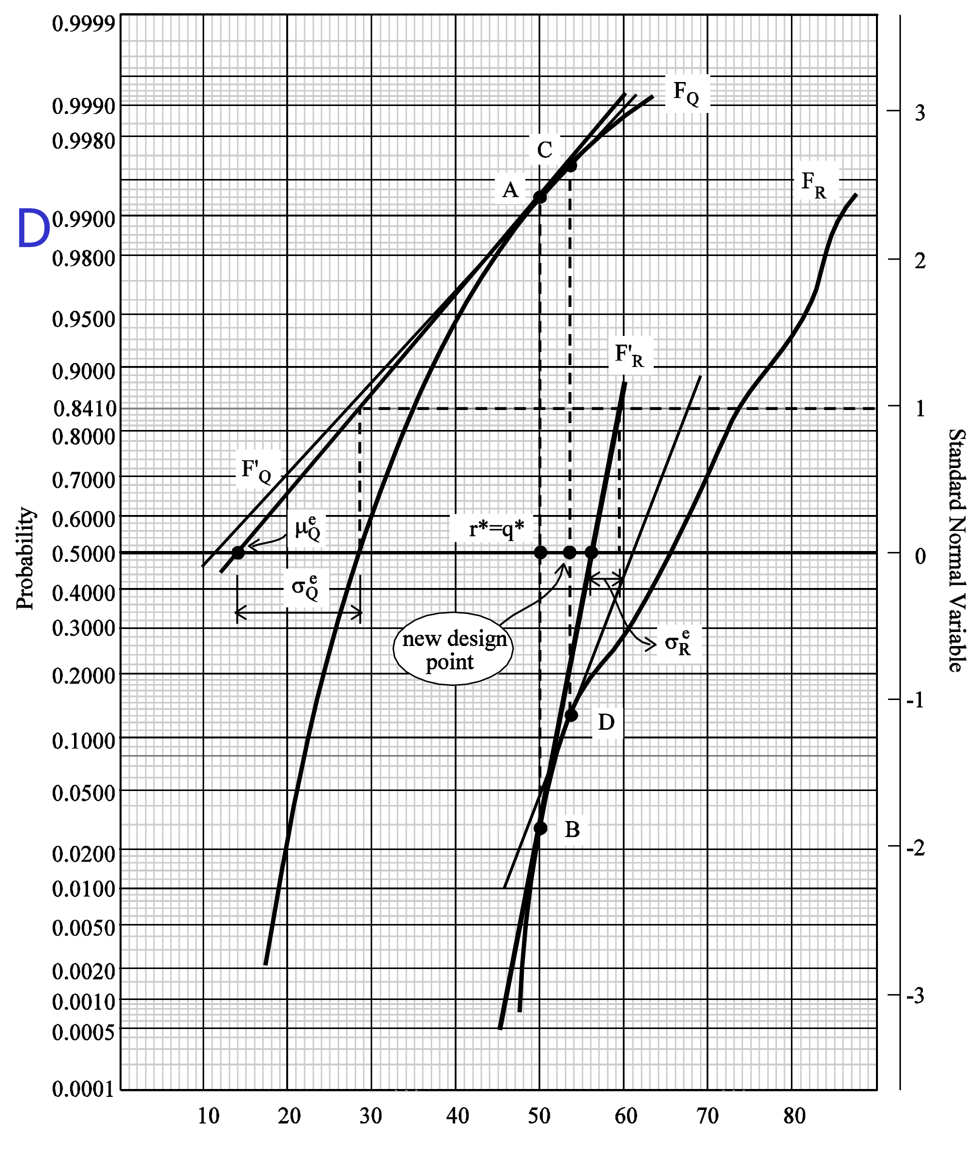


Figure 5-23 Graphical solution for Example Problem 5-12.

RACKWITZ-FIESSLER PROCEDURE
CORRELATED RANDOM VARIABLES

- So far, we have considered limit state equations in which the random variables are all uncorrelated.

- However, in many practical applications, some of the random variables may be correlated, and this correlation can have a significant impact on The calculated reliability index.

- To deal with correlated random variables, we can take two approaches:

RACKWITZ-FIESSLER PROCEDURE CORRELATED RANDOM VARIABLES

1. Use a coordinate transformation. This approach can become messy when dealing with the Rackwitz-Fiessler procedure involving equivalent normal parameters.

2. Modify the procedure presented by introducing a correlation matrix [ρ]. The correlation matrix [ρ] is the matrix of correlation coefficients for the random variables involved in the limit state equation. The modified equations are as follows:

$$\beta = \frac{\{G\}^T\{z^*\}}{\sqrt{\{G\}^T\{G\}}} \quad \text{changes to} \quad \beta = \frac{\{G\}^T\{z^*\}}{\sqrt{\{G\}^T[\rho]\{G\}}}$$

$$\{\alpha\} = \frac{\{G\}}{\sqrt{\{G\}^T\{G\}}} \quad \text{changes to} \quad \{\alpha\} = \frac{[\rho]\{G\}}{\sqrt{\{G\}^T[\rho]\{G\}}}$$

EXAMPLE

- Calculate the reliability index, β, for the limit state function

$$g(X_1, X_2) = 3X_1 - 2X_2$$

$$\mu_{X_1} = 16.6 \quad ; \quad \sigma_{X_1} = 2.45$$

$$\mu_{X_2} = 18.8 \quad ; \quad \sigma_{X_2} = 2.83$$

$$\text{Cov}(X_1, X_2) = 2.0.$$

We don't have any information on the distributions of X_1 and X_2, so we will assume they are both normally distributed.

EXAMPLE (continued)

1. Formulate limit state function and probability distributions.
2. Guess an initial design point. We will assume a value for x_1* of 17. From g = 0, x_2*= 25.5.
3. Determining equivalent normal parameters is not necessary since we are assuming both variables are normally distributed
4. Determine the values of the reduced variables

$$z_1^* = 0.163 \quad ; \quad z_2^* = 2.37$$

5. Determine the {G} vector

$$G_1 = -\left.\frac{\partial g}{\partial Z_1}\right|_{\{z_i^*\}} = -\left.\frac{\partial g}{\partial X_1}\right|_{\{x_i^*\}} \sigma_{X_1} = (-3)\sigma_{X_1}$$

$$G_2 = -\left.\frac{\partial g}{\partial Z_2}\right|_{\{z_i^*\}} = -\left.\frac{\partial g}{\partial X_2}\right|_{\{x_i^*\}} \sigma_{X_2} = 2\sigma_{X_2}$$

EXAMPLE (continued)

6. Calculate an estimate of β

$$[\rho] = \begin{bmatrix} 1 & \dfrac{Cov(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}} \\ \dfrac{Cov(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}} & 1 \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{2}{(2.45)(2.83)} \\ \dfrac{2}{(2.45)(2.83)} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.288 \\ 0.288 & 1 \end{bmatrix}$$

7. Calculate $\{\alpha\}$:

$$\beta = \frac{\{G\}^T\{z^*\}}{\sqrt{\{G\}^T[\rho]\{G\}}} = 1.55$$

$$\{\alpha\} = \frac{[\rho]\{G\}}{\sqrt{\{G\}^T[\rho]\{G\}}} = \begin{Bmatrix} -0.726 \\ 0.449 \end{Bmatrix}$$

EXAMPLE (continued)

8. Determine new values of z_i^* for n-1 of the variables

$$z_1^* = \alpha_1 \beta = (-0.726)(1.55) = -1.13$$

9. Determine the value of x_1^* using the value of z_1^*:

$$x_1^* = \mu_{X_1} + z_1^* \sigma_{X_1} = 13.8$$

10. Determine the value of x_2^* using the limit state equation g = 0.

x_2^*= 20.7

11. Iterate until the value of β and the design point converge.

EXAMPLE (continued)

Results (correct answer after one iteration)

	Iteration # 1	2
x_1^*	17	13.8
x_2^*	25.5	20.7
↓		
β	1.55	1.55
↓		
x_1^*	13.8	13.8
x_2^*	20.7	20.7

Reliability Analysis using Monte Carlo Method

- Given limit state function:

$$g = g\ (x_1,\ x_2,\ \ldots x_n)$$

where X_1, X_2, …X_n are random variables

- for each X_i, CDF is $F_{xi}\ (x)$

Reliability Analysis Procedure

1. For each random variable X_1, X_2 , …, X_n generate values of x_1, x_2 , …, x_n

2. Calculate

$$g = g\,(x_1, x_2, \ldots x_n)$$

3. Repeat steps 1 and 2, M times, resulting in M values of g: g_1, g_2, …g_M

4. Determine the probability of failure, P_f, and reliability index, β

Probability of Failure, P_f, and Reliability Index, β

<u>Option (a)</u>

Count the number, m, of negative values of g,

calculate

$$P_f = m/M$$

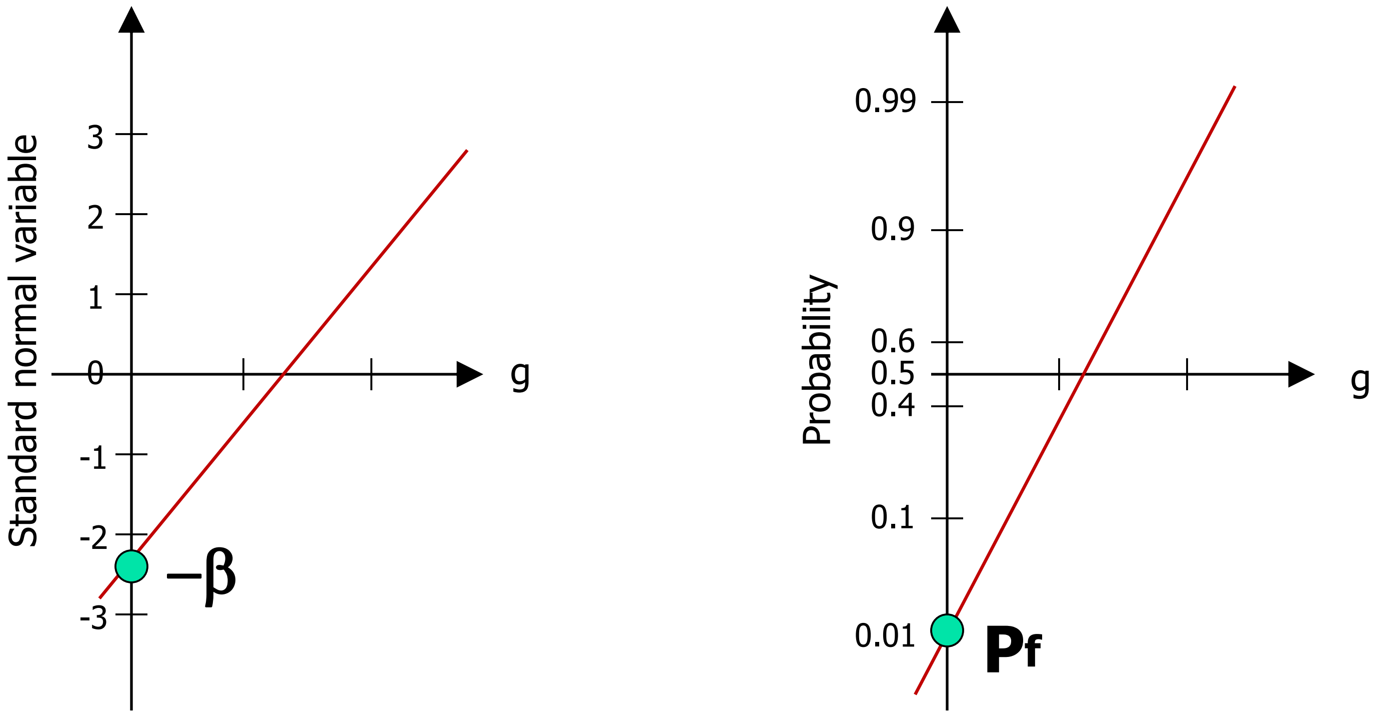
$$\beta = -\Phi^{-1}(P_f)$$

For accurate results, m should be ≥ 10

Probability of Failure, P_f, and Reliability Index, β

Option (b)

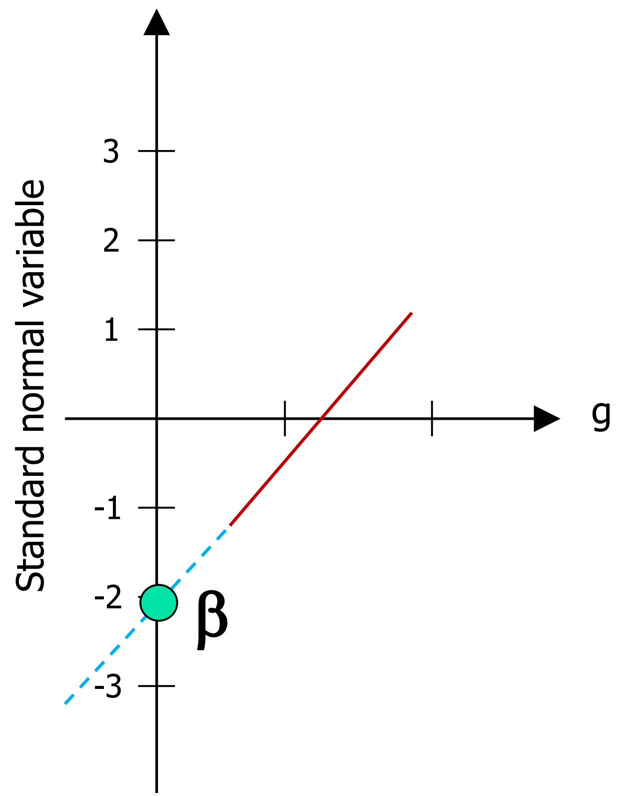
Plot CDF of g on the normal probability paper



Probability of Failure, P_f, and Reliability Index, β

Option (c)

If CDF of g is too short then either increase M or extrapolate



Reliability Analysis using Monte Carlo Method - Example

- Calculate P_f and β
- Given limit state function:

$$g\,(R,D,L) = R - (D + L)$$

Reliability Analysis using Monte Carlo Method - Example

- Given:
  - Dead load, normally distributed
    - D = 75 k-ft
    - λ_D = 1.03
    - v_D = 0.10
  - Live load, normally distributed
    - L = 110 k-ft
    - λ_L = 0.80
    - v_L = 0.14
  - Resistance, lognormally distributed
    - L = 255 k-ft
    - λ_R = 1.11
    - v_R = 0.12

- Assume D, L, R are uncorrelated random variables

Calculate mean values and standard deviations

- Dead load

$\mu_D = D\ \lambda_D = 77.25$ k-ft

$\sigma_D = \mu_D\ V_D = 7.73$ k-ft

- Live load

$\mu_L = L\ \lambda_L = 88.00$ k-ft

$\sigma_L = \mu_L\ V_L = 12.32$ k-ft

- Resistance

$\mu_R = R\ \lambda_R = 283.05$ k-ft

$\sigma_R = \mu_L\ V_R = 33.97$ k-ft

$\mu_{ln(R)} = ln(\mu_X) = 5.65$ k-ft

$\sigma_{ln(X)}^2 = V_R^2 = 0.0144$

$\sigma_{ln(X)} = 0.12$ k-ft

Monte Carlo Simulation Results

(18 out of 500 runs in this example)

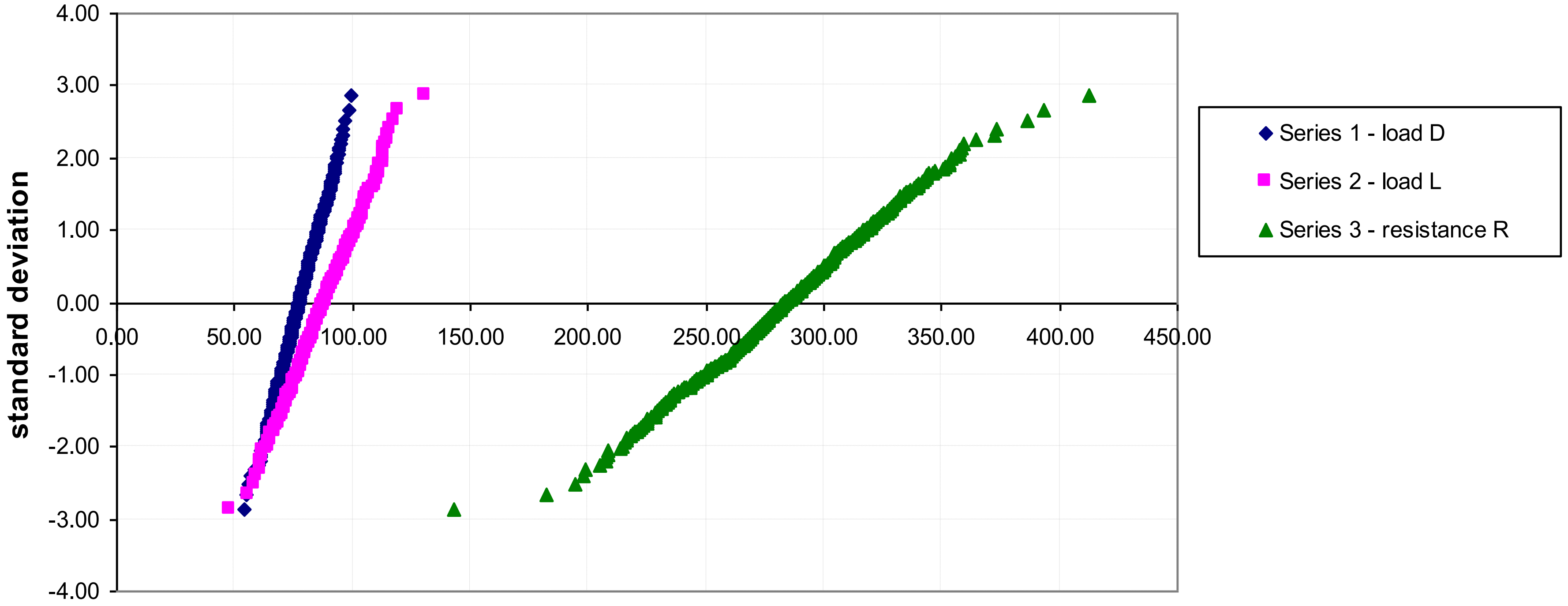
no	random	z_i	D_i	random	z_i	L_i	random	z_i	R_i	R-D-L
1	0.51938	0.0486	77.63	0.66292	0.4204	93.18	0.70252	0.5317	301.11	130.30
2	0.96848	1.8589	91.61	0.28507	-0.5678	81.00	0.99603	2.6544	373.21	200.60
3	0.58211	0.2073	78.85	0.47295	-0.0678	87.16	0.90875	1.3331	328.33	162.31
4	0.73213	0.6193	82.03	0.56859	0.1728	90.13	0.01083	-2.2962	205.06	32.90
5	0.24471	-0.6912	71.91	0.30698	-0.5044	81.79	0.56467	0.1628	288.58	134.88
6	0.29664	-0.5341	73.12	0.29250	-0.5461	81.27	0.73470	0.6271	304.35	149.95
7	0.80078	0.8444	83.77	0.19941	-0.8437	77.61	0.09938	-1.2851	239.40	78.02
8	0.23012	-0.7384	71.55	0.78669	0.7950	97.79	0.51991	0.0499	284.75	115.41
9	0.39553	-0.2649	75.20	0.63161	0.3361	92.14	0.58271	0.2088	290.14	122.80
10	0.12148	-1.1676	68.23	0.74777	0.6675	96.22	0.01444	-2.1852	208.83	44.37
11	0.71842	0.5782	81.72	0.16179	-0.9871	75.84	0.35544	-0.3707	270.46	112.90
12	0.18739	-0.8875	70.39	0.48303	-0.0426	87.48	0.74109	0.6467	305.02	147.15
13	0.16641	-0.9684	69.77	0.97407	1.9444	111.95	0.16753	-0.9640	250.31	68.58
14	0.45987	-0.1008	76.47	0.40025	-0.2527	84.89	0.72532	0.5987	303.39	142.03
15	0.43911	-0.1532	76.07	0.19361	-0.8647	77.35	0.59049	0.2288	290.82	137.41
16	0.83112	0.9586	84.66	0.61738	0.2986	91.68	0.90234	1.2950	327.04	150.70
17	0.33991	-0.4127	74.06	0.07903	-1.4116	70.61	0.69240	0.5027	300.12	155.45
18	0.32084	-0.4653	73.66	0.82758	0.9447	99.64	0.67419	0.4515	298.39	125.09

Monte Carlo Simulation Results
(18 out of 500 runs in this example)

no	R-D-L	M/(M+1)	Φ^{-1}[M/(M+1)]
1	-25.51	0.0020	-2.8788
2	21.90	0.0040	-2.6527
3	28.46	0.0060	-2.5128
4	28.47	0.0080	-2.4096
5	30.70	0.0100	-2.3271
6	32.90	0.0120	-2.2579
7	33.58	0.0140	-2.1981
8	34.09	0.0160	-2.1452
9	39.89	0.0180	-2.0977
10	41.45	0.0200	-2.0546
11	43.71	0.0220	-2.0149
12	44.37	0.0240	-1.9782
13	44.63	0.0259	-1.9440
14	46.55	0.0279	-1.9119
15	47.30	0.0299	-1.8817
16	50.16	0.0319	-1.8531
17	50.40	0.0339	-1.8259
18	50.87	0.0359	-1.8000

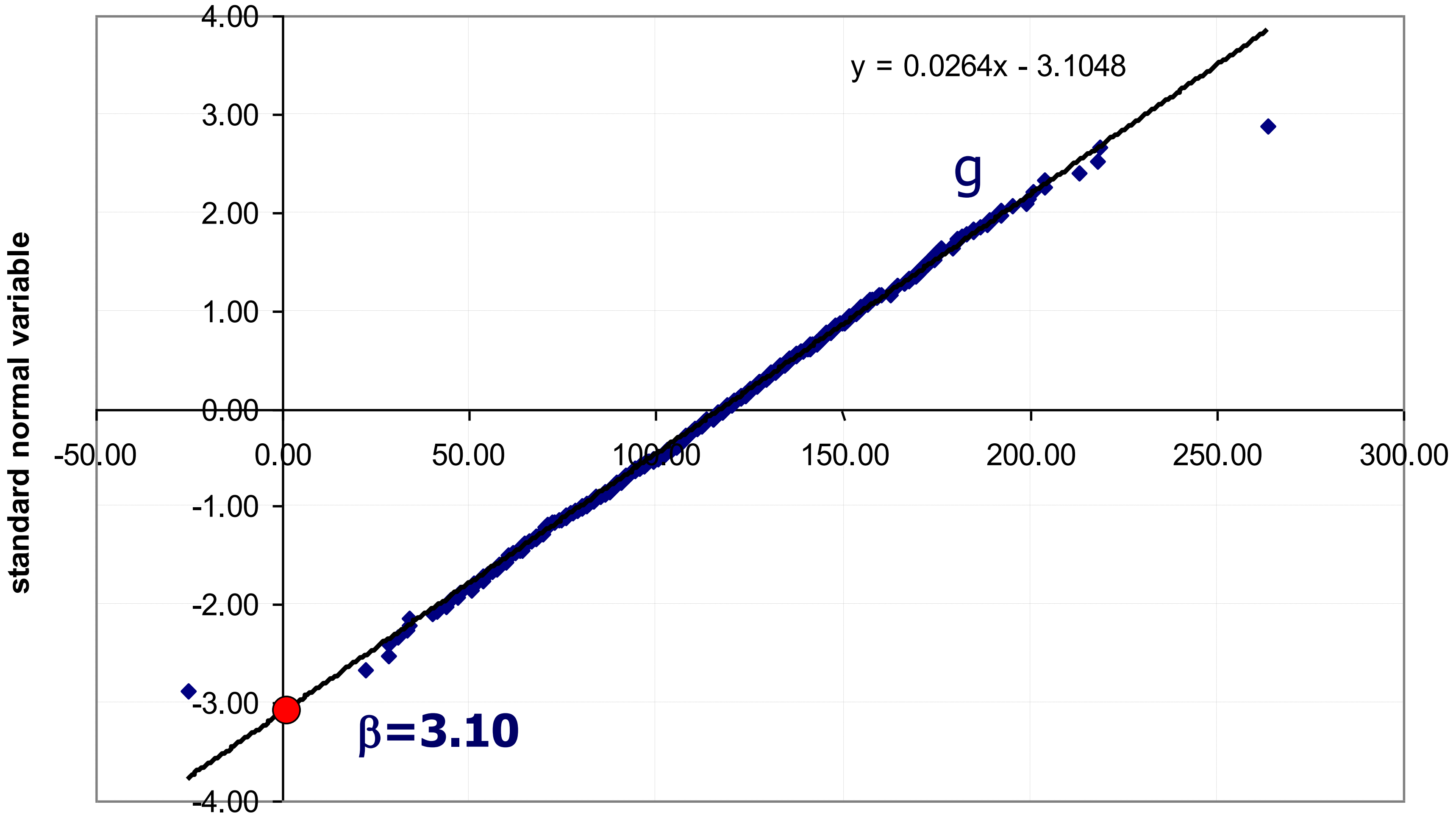
Results of Simulation: CDF's of D, L and R

(500 runs)



Results of Simulation, g = R-D-L

(500 runs)



Results of Simulation: P_f and β

(500 runs)

Reliability index

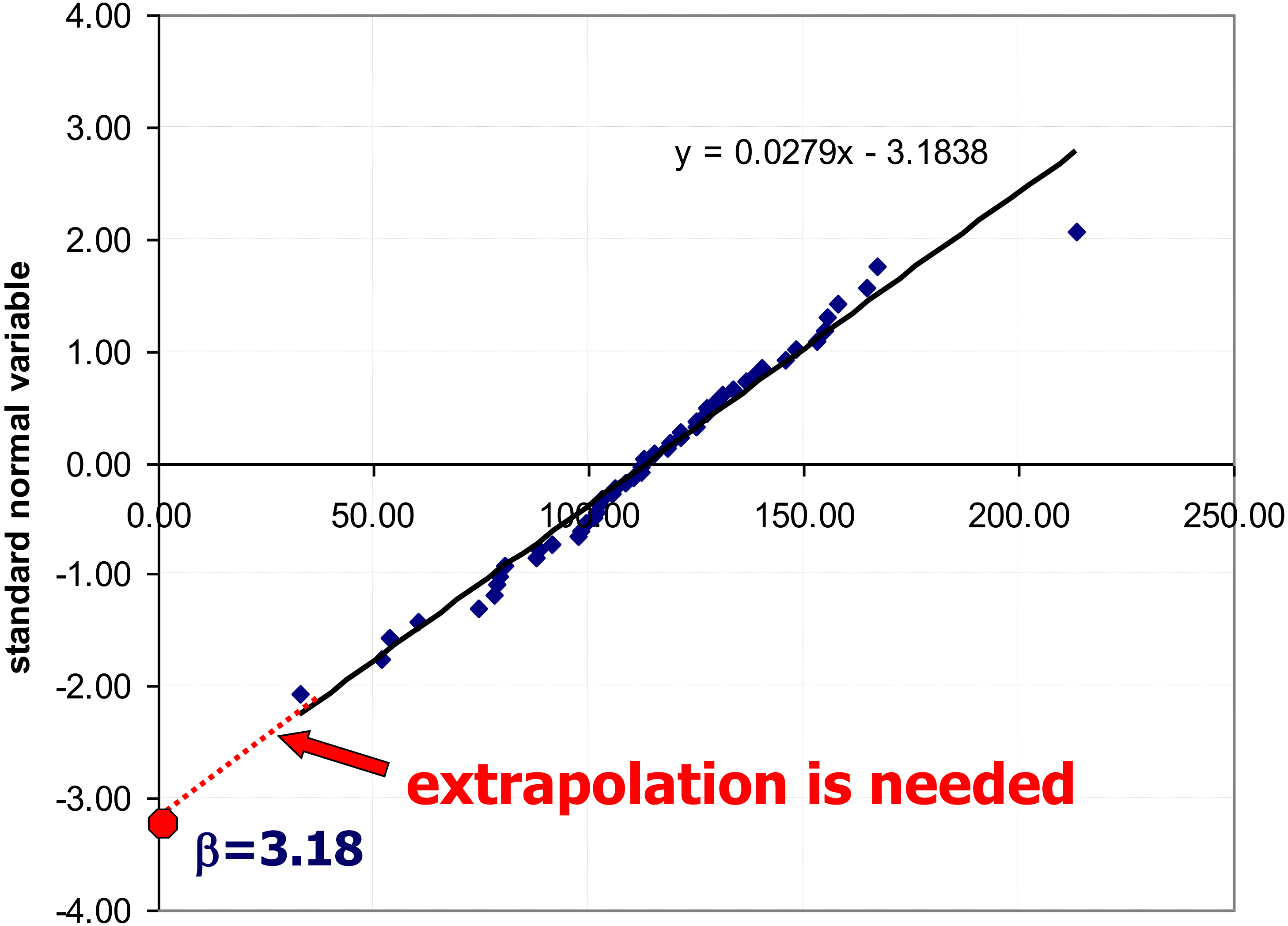
$$\beta = 3.10$$

Probability of failure

$$P_f = \Phi(-\beta) = 0.0010$$

Results of Simulation, g = R-D-L

(50 runs)



Results of Simulation: P_f and β

(50 runs)

Reliability index

$$\beta = 3.18$$

Probability of failure

$$P_f = \Phi(-\beta) = 0.0007$$