Reliability of Structures – Part 6

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System Reliability

- Elements
- Systems
- Effect of Correlation

SYSTEM RELIABILITY

- Elements and Systems
- Series System
- Parallel System
- Reliability Bounds for Structural Systems
- Systems with Equally Correlated Elements
  - Series System
  - Parallel System with Ductile Elements
- System with Unequally Correlated Elements
  - Parallel System
  - Series System
- Summary

ELEMENTS AND SYSTEMS

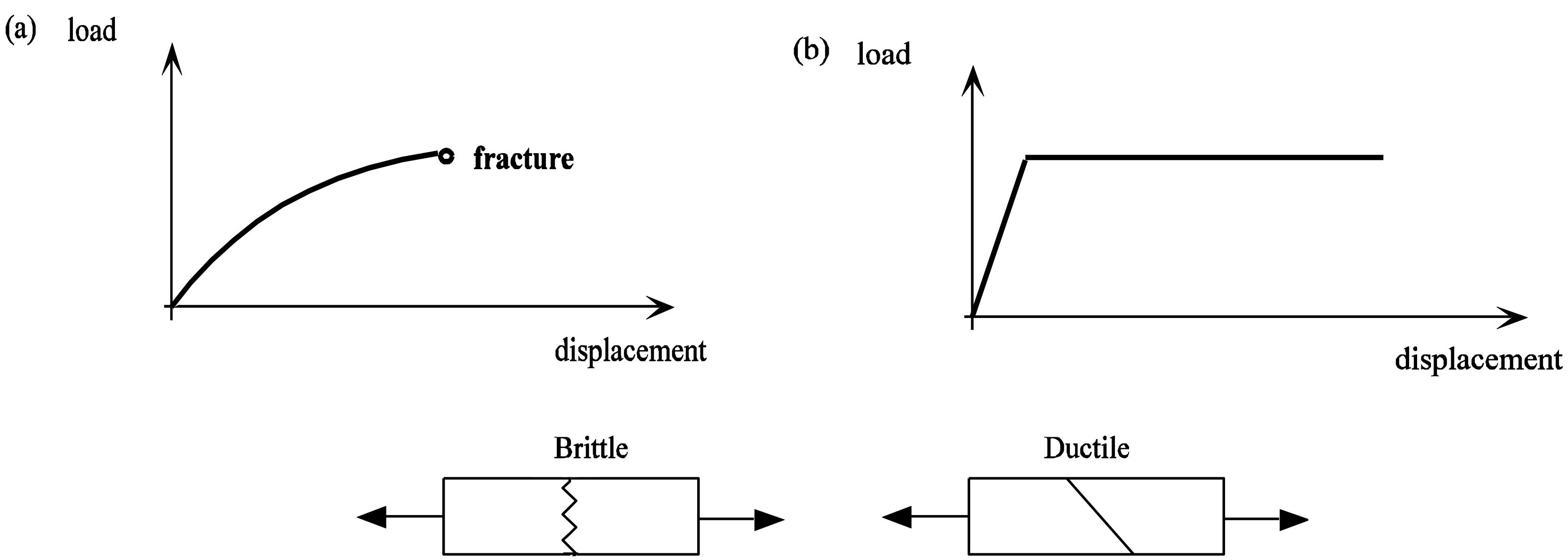
The system is a system of interconnected components.

In general, failure of a single component does not mean failure of the structure.

Failure of a component can mean reaching the limit state (e.g. ultimate load carrying capacity)

ELEMENTS AND SYSTEMS

- Brittle Elements – a structure is perfectly brittle if it becomes ineffective after failure
- Ductile Elements – a structure is perfectly ductile if it maintains its load carrying capacity after failure.



SERIES SYSTEMS

- A system of single elements is a series system if it is in a state of failure whenever any of its elements fails. Such a system is also called a weakest-link system

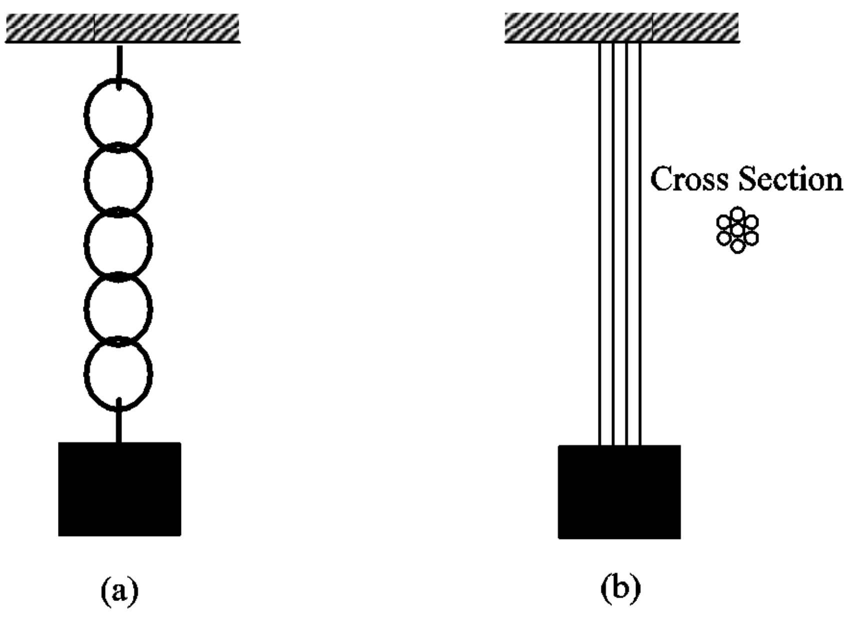
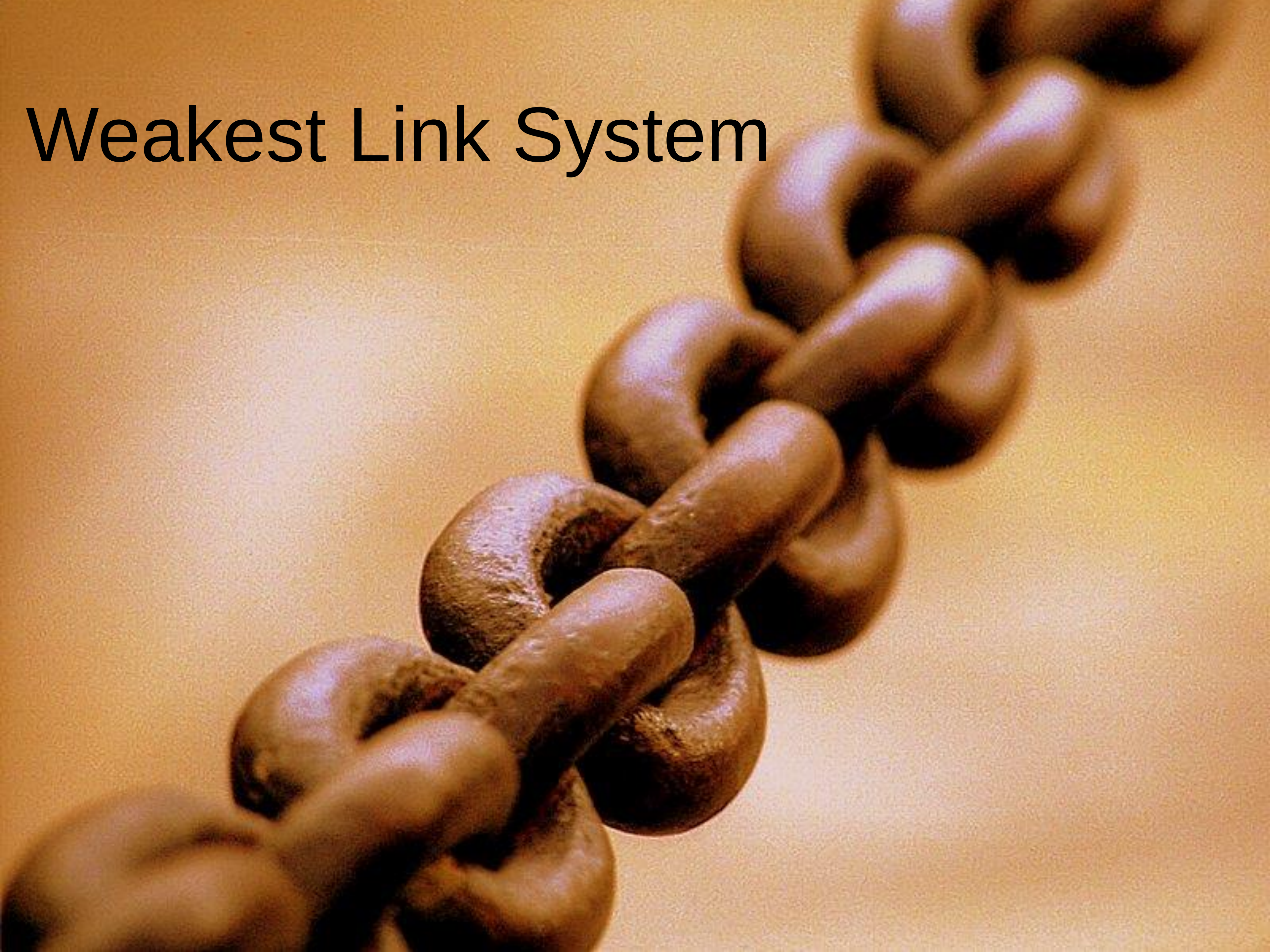


Figure 9-3 (a) Example of a Series System. (b) Example of a Parallel System.

Weakest Link System



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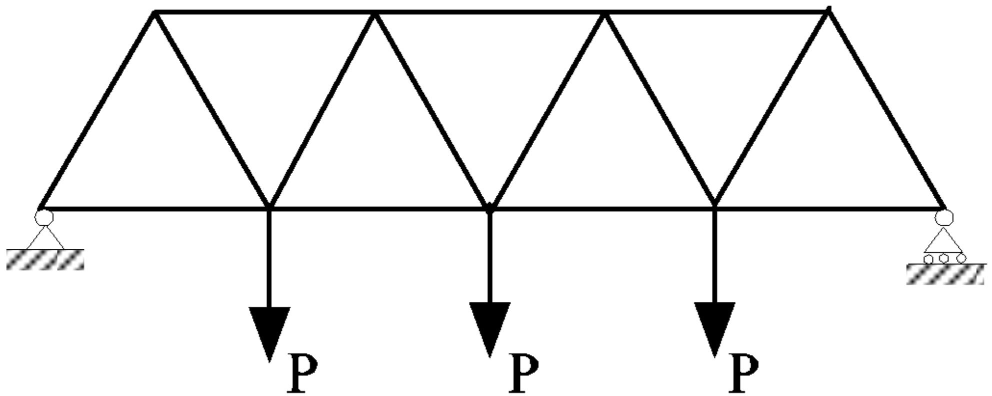
SERIES SYSTEMS

- An example of a series system



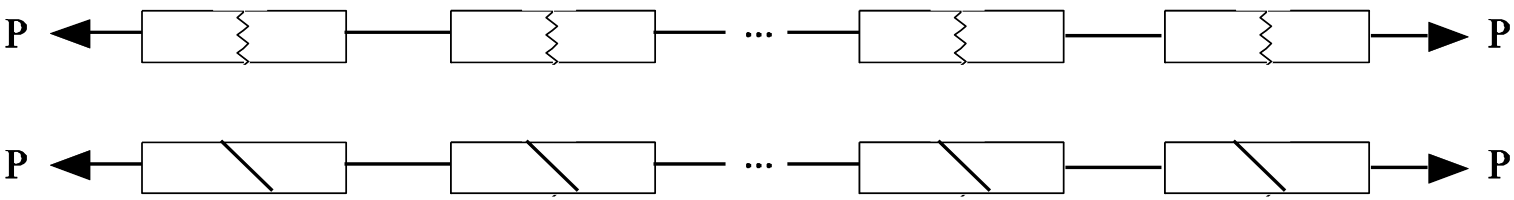


Figure 9-5 Examples of series systems using the symbols shown in Figure 9-2.


AP

SERIES SYSTEMS

- Failure of the system implies that the strength R is less than the load q. In terms of probability, the probability of failure, P_f, would be

$$P_f = P(R < q) = F_R(q)$$

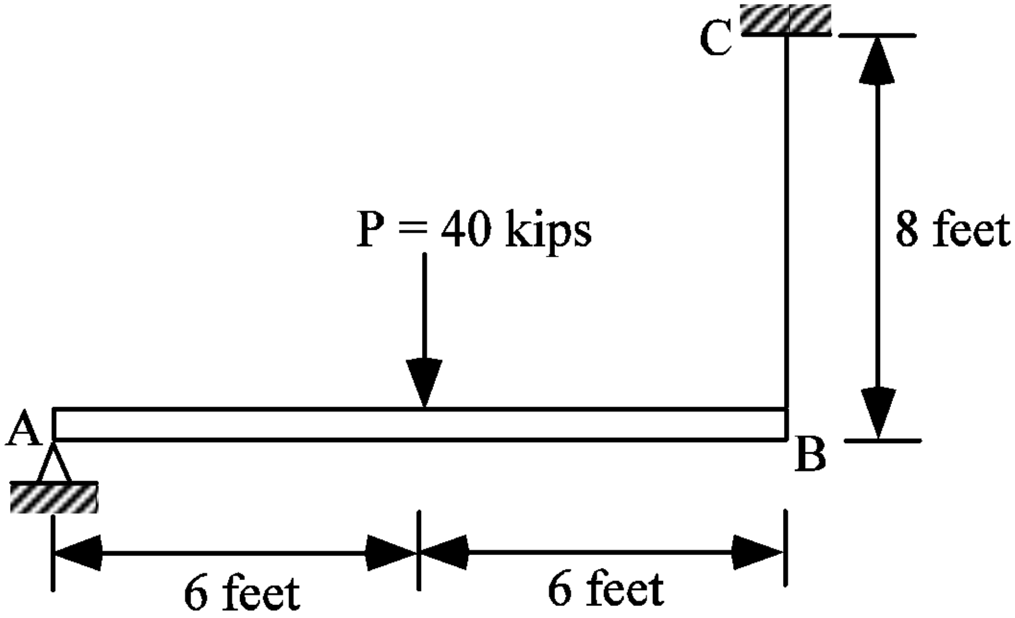
- With this interpretation, and assuming that the strengths of the elements are all statistically independent, we can calculate the probability of failure as:

SERIES SYSTEMS

$$
\begin{aligned}
P_f &= F_R(q) \\
&= P(R < q) \\
&= 1 - P(R \geq q) \\
&= 1 - P\left[(R_1 \geq q_1) \cap (R_2 \geq q_2) \cap \ldots \cap (R_n \geq q_n)\right] \\
&= 1 - P(R_1 \geq q_1) P(R_2 \geq q_2) \ldots\ldots P(R_n \geq q_n) \\
&= 1 - \left[1 - P(R_1 < q_1)\right]\left[1 - P(R_2 < q_2)\right] \ldots\ldots \left[1 - P(R_n < q_n)\right] \\
&= 1 - \prod_{i=1}^{n} \left[1 - F_{R_i}(q_i)\right] \\
&= 1 - \prod_{i=1}^{n} \left[1 - P_{f_i}\right]
\end{aligned}
$$

SERIES SYSTEMS

Example



Member AB is a steel beam (W10x54) with a yield stress F_y = 36 ksi and a plastic section modulus Z = 66.6 in3. Member BC is a steel bar with diameter D = 1 inch and an ultimate strength of F_u = 58 ksi.

Calculate the reliability of the system.

SERIES SYSTEMS - example

The following statistical parameters are assumed for the resistance of the members, R:

For R_{AB}: $\lambda_{AB} = 1.07$ $V_{AB} = 0.13$ R_{AB} is lognormal

For R_{BC}: $\lambda_{BC} = 1.14$ $V_{BC} = 0.14$ R_{BC} is normal

The resistance of the beam is based on its nominal plastic moment capacity

$$R_n \text{ for AB} = Z\,F_y = (66.6)(36) = 2398 \text{ kip-in}$$

SERIES SYSTEMS - example

Therefore, the mean value and standard deviation of the resistance of AB are

$$\mu_{R_{AB}} = \lambda_{AB} R_n = (1.07)(2398) = 2566$$

$$\sigma_{R_{AB}} = \mu_{R_{AB}} V_{AB} = (2566)(0.13) = 334$$

Since R_{AB} follows a lognormal distribution, the relevant distribution parameters are $\mu_{\ln R}$ and $\sigma_{\ln R}$.

$$\sigma_{\ln R} = \sqrt{\ln\left(1 + V_{AB}^2\right)} = 0.129$$

$$\mu_{\ln R} = \ln(\mu_{R_{AB}}) - 0.5\sigma_{\ln R}^2 = 7.84$$

SERIES SYSTEMS - example

For the steel bar, the nominal resistance is the cross-section area multiplied by the ultimate stress:

$$R_n \text{ for BC} = AF_u = \frac{\pi}{4} D^2 F_u = \frac{\pi}{4}(1)^2(58) = 45.6 \text{ kips}$$

Therefore, the mean value and standard deviation of the resistance of BC are

$$\mu_{R_{BC}} = \lambda_{BC} R_n = (1.14)(45.6) = 52.0\text{kips}$$

$$\sigma_{R_{BC}} = \mu_{R_{BC}} V_{BC} = (52.0)(0.14) = 7.28\text{kips}$$

SERIES SYSTEMS - example

Now, we must evaluate how to define failure of the elements and the overall system. Based on a static analysis of the system, the vertical reaction at A and the force in the cable are both 0.5P = 20 kips. The beam AB will fail when the moment capacity is less than the maximum moment which is

$$M_{max} = \frac{PL}{4} = \frac{(40\text{ kips})(12\text{ feet})}{4} = 120\text{ kip-ft} = 1{,}440\text{ kip-in}$$

Therefore, the probability of failure of member AB is

$$P_f\ for AB = P(R_{AB} < 1{,}440) = \Phi\left(\frac{\ln(1{,}440) - \mu_{\ln R}}{\sigma_{\ln R}}\right) = \Phi(-4.40) = 5.41 \times 10^{-6}$$

SERIES SYSTEMS - example

The steel bar will fail when its capacity is less than 0.5P = 20 kips,

$$P_f \text{ for BC} = P(R_{BC} < 20) = \Phi\left(\frac{20-\mu_{R_{BC}}}{\sigma_{R_{BC}}}\right) = \Phi(-4.39) = 5.67 \text{ x } 10^{-6}$$

The overall system will fail if either the beam or the bar fails.

$$P_f = 1 - \prod_{i=1}^{2}\left[1 - P_{f_i}\right]$$

$$= 1 - \left[1 - \left(5.41\text{x}10^{-6}\right)\right]\left[1 - \left(5.67\text{x}10^{-6}\right)\right]$$

$$= 1.11\text{x}10^{-5}$$

SERIES SYSTEMS - example

In the reliability studies, it is often more convenient to convert probabilities of failure to reliability indices and then to compare the indices.

$$\beta_{AB} = -\Phi^{-1}(P_f \; for AB) = -\Phi^{-1}(5.41x10^{-6}) = 4.40$$

$$\beta_{BC} = -\Phi^{-1}(P_f \; for BC) = -\Phi^{-1}(5.67x10^{-6}) = 4.39$$

$$\beta_{\text{system}} = -\Phi^{-1}(P_f \, \text{for system}) = -\Phi^{-1}(1.11x10^{-5}) = 4.24$$

Observe that β_{AB} and β_{BC} are larger than the reliability index for the entire system. This is a common characteristic of a series system. The lower value of β_{system} tells us that the system is “less reliable” than either of its components.

SERIES SYSTEMS – example 2

Consider a series system with n elements. If the probability of failure for each element is 0.05, then the probability of failure for the system, P_f, is

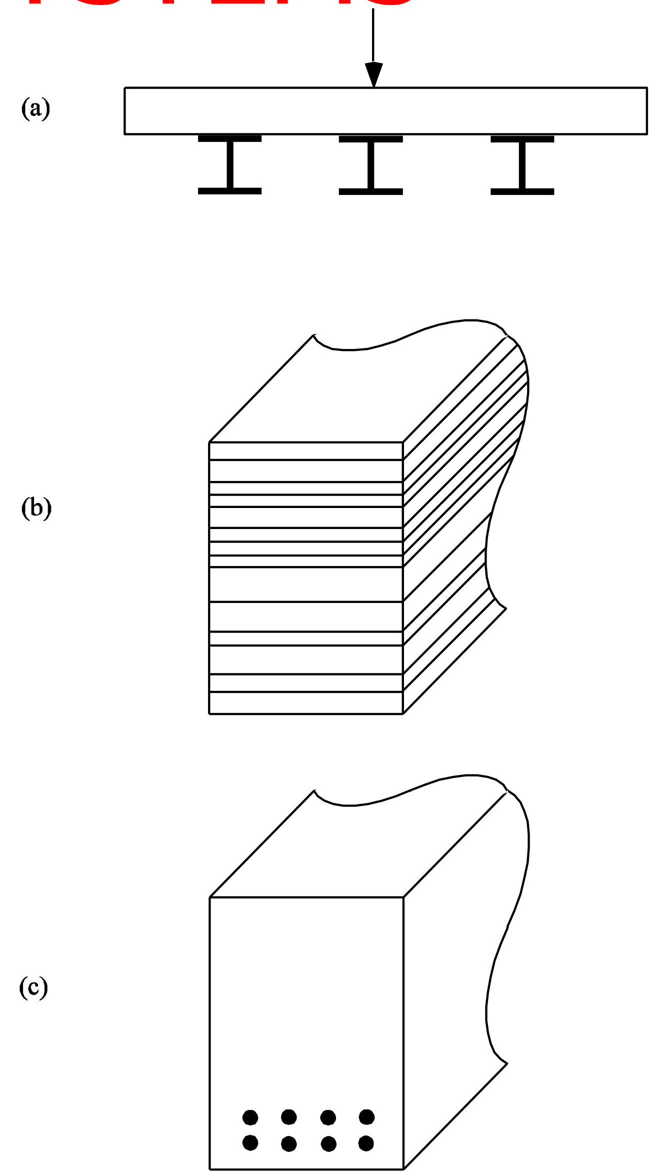
$$P_f = 1 - \prod_{i=1}^{n} \left[1 - P_{f_i} \right]$$

$$= 1 - \left[1 - 0.05 \right]^n$$

$$= 1 - \left[0.95 \right]^n$$

n	P_f	$\beta = -\Phi^{-1}(P_f)$
1	0.05	1.64
2	0.0975	1.295
3	0.1426	1.07
5	0.2262	0.75
10	0.4013	0.25

PARALLEL SYSTEMS



A parallel system can consist of ductile or brittle elements.

These systems qualify as parallel systems because failure of all components is required for the overall system to fail.

Parallel System

Cables in Brooklyn Bridge

Golden Gate Bridge, San Francisco
built in 1933-1935, span of 1280 m


Golden Gate Bridge
San Francisco

PARALLEL SYSTEMS with Perfectly Ductile Elements

A parallel system with n *perfectly ductile* elements is in a state of failure when all of its elements fail (i.e., yield). Let R_i represent the strength of the i^{th} element in such a system. The system strength will be the sum of all the strengths of the elements, or

$$R = \sum_{i=1}^{n} R_i$$

If the strengths of the individual elements are all *uncorrelated* variables, then

$$\mu_R = \sum_{i=1}^{n} \mu_{R_i} \qquad \sigma_R^2 = \sum_{i=1}^{n} \sigma_{R_i}^2$$

PARALLEL SYSTEMS with Perfectly Ductile Elements

The probability of failure of the system can be determined as follows.

$$
\begin{aligned}
P_f &= P\left[(R_1 < q_1) \cap (R_2 < q_2) \cap \ldots \cap (R_n < q_n)\right] \\
&= P(R_1 < q_1) P(R_2 < q_2) \ldots\ldots P(R_n < q_n) \\
&= F_{R_1}(q_1) F_{R_2}(q_2) \ldots\ldots F_{R_n}(q_n) \\
&= \prod_{i=1}^{n} F_{R_i}(q_i) \\
&= \prod_{i=1}^{n} P_{f_i}
\end{aligned}
$$

PARALLEL SYSTEMS with Perfectly Ductile Elements

Now, consider a special case of this same system of uncorrelated, perfectly ductile elements. If the R_i variables are identically distributed (meaning that they have the same PDF and CDF), then the mean and variance become

$$\mu_R = n\,\mu_{R_i} \qquad\qquad \sigma_R^2 = n\,\sigma_{R_i}^2$$

The corresponding coefficient of variation is

$$V_R = \frac{\sigma_R}{\mu_R} = \frac{\sqrt{n\,\sigma_{R_i}^2}}{n\,\mu_{R_i}} = \frac{1}{\sqrt{n}}\,\frac{\sigma_{R_i}}{\mu_{R_i}} = \frac{1}{\sqrt{n}}\,V_{R_i}$$

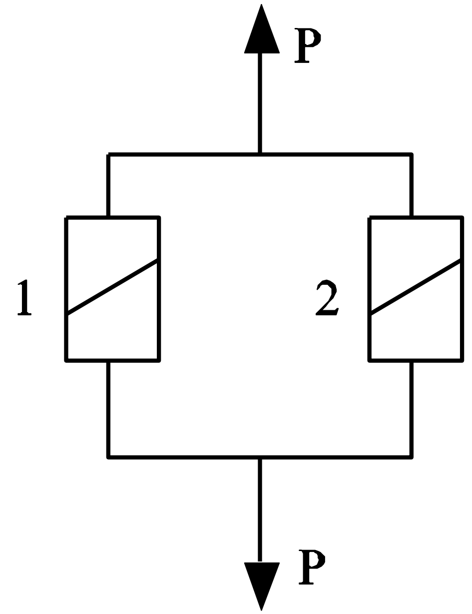
PARALLEL SYSTEMS with Perfectly Ductile Elements

Example

Consider a parallel system consisting of two perfectly ductile elements. The strengths of both elements are normally distributed with the following parameters

$$\mu_{R_1} = \mu_{R_2} = 5kN$$

$$V_{R_1} = V_{R_2} = 0.20$$



PARALLEL SYSTEMS with Perfectly Ductile Elements

The standard deviation of the strength of each element is simply

$$\sigma_{R_1} = \mu_{R_1} V_{R_1} = 1\,\mathrm{kN} = \sigma_{R_2}$$

Since both elements are identically distributed, the result is

$$\mu_R = n\,\mu_{R_i} = 10\,\mathrm{kN}$$

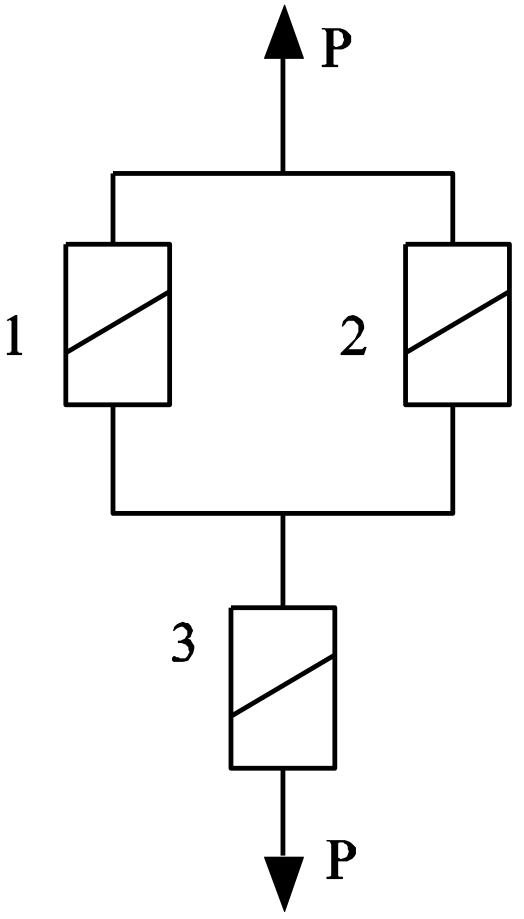
$$\sigma_R^2 = n\,\sigma_{R_i}^2 = 2\,\mathrm{kN} \quad \Rightarrow \quad \sigma_R = \sqrt{2}\,\mathrm{kN}$$

$$V_R = \frac{1}{\sqrt{n}} V_{R_i} = \frac{1}{\sqrt{2}}(0.20) = 0.14$$

HYBRID (Combined) SYSTEMS

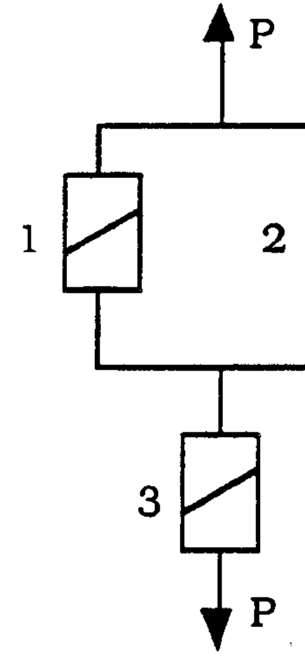
Many actual structures can be considered as a combination of series and parallel systems. Such systems are referred to as hybrid or combined systems.

In the example, elements 1 and 2 are in parallel, and the combination of 1 and 2 is in series with element 3.



SYSTEM RELIABILITY EXAMPLE

Strengths are R_1, R_2, R_3. R_1 and R_2 are normally distributed and R_3 is a uniformly distributed random variable. All are uncorrelated random variables.



The parameters of R_1 and R_2 are,

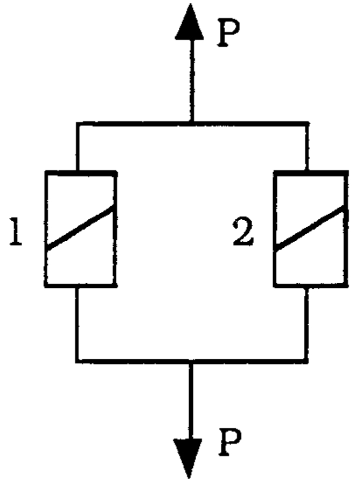
$$\overline{R_1} = \overline{R_2} = 5 \text{ kN}; \quad \sigma_{R_1} = \sigma_{R_2} = 1 \text{ kN}$$

R_3 is uniform over the interval (8, 12 kN).

SYSTEM RELIABILITY EXAMPLE

Consider first a sub-system made of elements 1 and 2. The strength of this sub-system is R_{12}. This sub-system is a parallel system . Therefore, the mean and the standard deviation can be calculated as follow:

The strength of sub-system with elements 1 and 2, R_{12}, is a normal random variable. The parameters of R_{12} are,



$$\overline{R}_{12} = \overline{R}_1 + \overline{R}_2 = 10 \text{ kN}$$

$$\sigma^2_{R_{12}} = \sigma^2_{R_1} + \sigma^2_{R_2} = 1 + 1 = 2 \text{ kN}$$

$$\sigma_{R_{12}} = \sqrt{2} \text{ kN}$$

SYSTEM RELIABILITY EXAMPLE

The CDF of the strength of the third element, R_3, is given by,

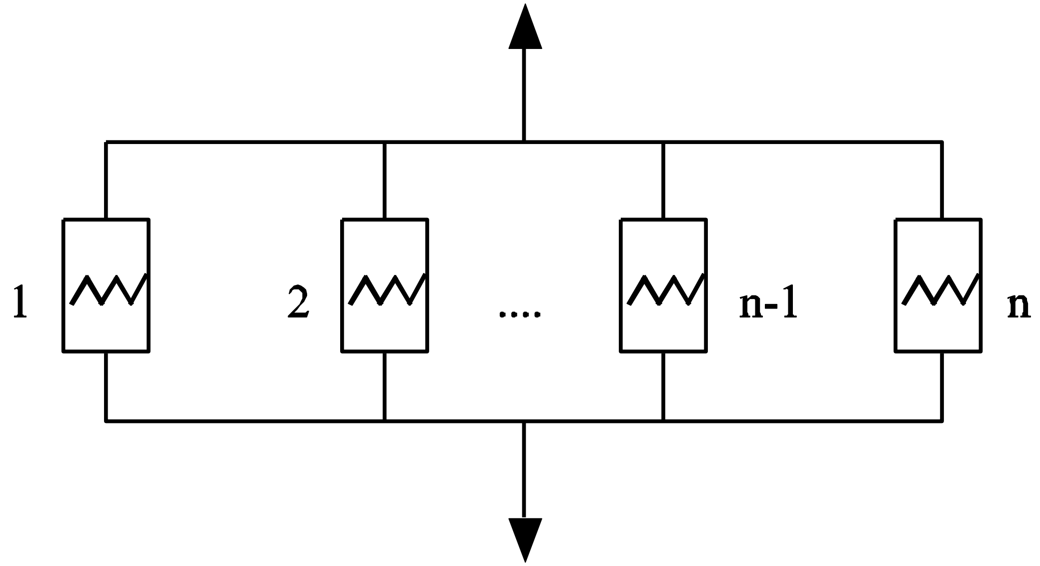
$$F_{R_3}(r) = \begin{cases} 0 & \text{for } r < 8 \\ \frac{1}{4} r - 2 & \text{for } 8 \le r < 12 \\ 1 & \text{for } r \ge 12 \end{cases}$$

Sub-system 1-2 and element 3 form a series system. Therefore, CDF can be calculated using Eq. 9-1.

$$F_R(r) = 1 - \prod_{i=1}^{n} \left(1 - F_{R_i}(r_i)\right) = 1 - \left(1 - F_{R_{12}}(r)\right)\left(1 - F_{R_3}(r)\right) =$$

$$= \Phi\left(\frac{r-10}{\sqrt{2}}\right) + F_{R_3}(r) - \Phi\left(\frac{r-10}{\sqrt{2}}\right) F_{R_3}(r)$$

PARALLEL SYSTEMS with Brittle Elements



Let R_1, R_2, ..., R_n represent the strengths of elements 1, 2,..., n. Furthermore, assume that the strengths are ordered such that $R_1 \leq R_2 \leq \ldots \leq R_n$. With these assumptions, the strength of the system, R, is

$$R = \max\left[nR_1, (n-1)R_2, (n-2)R_3, \ldots, 2R_{n-1}, R_n\right]$$

Boolean Variables

Assumption: each element can exist in only one of two states

- failure
- non-failure (safe)

Boolean variables, S_i and F_i:

$$S_i = \begin{cases} 1 & \text{if the element is in a nonfailure state} \\ 0 & \text{if the element is in a failure state} \end{cases}$$

$$F_i = 1 - S_i = \begin{cases} 0 & \text{if the element is in a nonfailure state} \\ 1 & \text{if the element is in a failure state} \end{cases}$$

Boolean Variables

State of the system is described by vector,

$$\overline{S} = \{S_1, S_2, \ldots, S_n\}$$

System function is defined as,

$$S_S(\overline{S}) = \begin{cases} 1 & \text{if the system is in a nonfailure state} \\ 0 & \text{if the system is in a failure state} \end{cases}$$

For a series system, failure of one element means system failure

$$S_S(\overline{S}) = S_1 S_2 \ldots S_n = \prod_{i=1}^{n} S_i$$

or

$$S_S(\overline{F}) = 1 - S_S(\overline{S}) = 1 - \prod_{i=1}^{n} (1 - F_i)$$

Boolean Variables

For a parallel system, if at least one element is in a non-failure state, then the system is in a non-failure state.

$$S_S(\overline{S}) = 1 - \prod_{i=1}^{n} (1 - S_i) \quad \text{or} \quad S_S(\overline{F}) = \prod_{i=1}^{n} (F_i)$$

So, if one element is in a non-failure state, then $(1 - S_i) = 0$ for that element and the product in equation

$$S_S(\overline{S}) = S_1 S_2 \ldots . S_n = \prod_{i=1}^{n} S_i$$

will be equal to zero. The resulting function will be equal to 1 (the system is in a non-failure state).

SERIES SYSTEMS with Positive Correlation

When the correlation coefficient ρ_{ij} is greater than or equal to zero the following bounds can be applied:

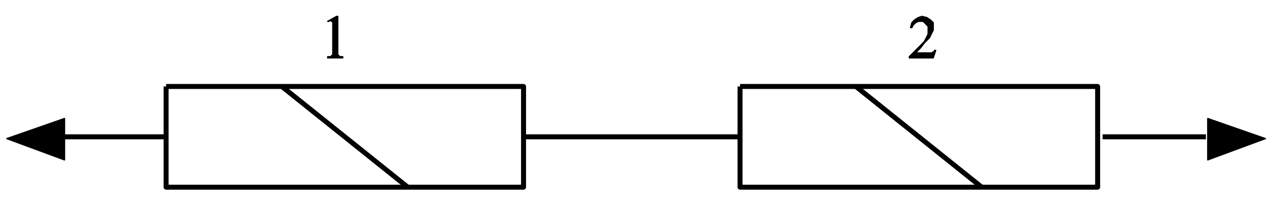
$$\max_i \left\{ P\left[F_i = 1\right] \right\} \leq P_f \leq 1 - \prod_{i=1}^{n} \left(1 - P\left[F_i = 1\right]\right)$$

all elements are fully correlated (ρ_{ij} = 1).

all elements are uncorrelated (ρ_{ij} = 0).

SERIES SYSTEMS with Positive Correlation

Example: Consider two perfectly ductile elements of a series system. Assume that the reliability index for each element (1 and 2) is $\beta_e = 3.5$. The strengths (resistance) of both elements are normally distributed. Determine the upper and lower bounds on the failure probability of the system.



SERIES SYSTEMS with Positive Correlation-example

Since the two elements are identically distributed, the lower bound is simply

$$\max_i \left\{ P\left[F_i = 1 \right] \right\} = \quad P(F_1=1)=P(F_2=1)= \Phi(-\beta_e)= \Phi(-3.5) = \underline{0.000233}$$

For the upper bound

$$1 - \prod_{i=1}^{n} \left(1 - P\left[F_i = 1 \right] \right) = 1 - (1 - 0.000233)^2 = \underline{0.000466}$$

The exact probability of failure of the system is between these bounds

$$0.000233 \leq P_f \leq 0.000466$$

PARALLEL SYSTEMS with Positive Correlation

The bounds on the probability of failure for a parallel system with positive correlation are as follows

$$\prod_{i=1}^{n} P[F_i = 1] \quad \leq \quad P_f \quad \leq \quad \min_i \{P[F_i = 1]\}$$

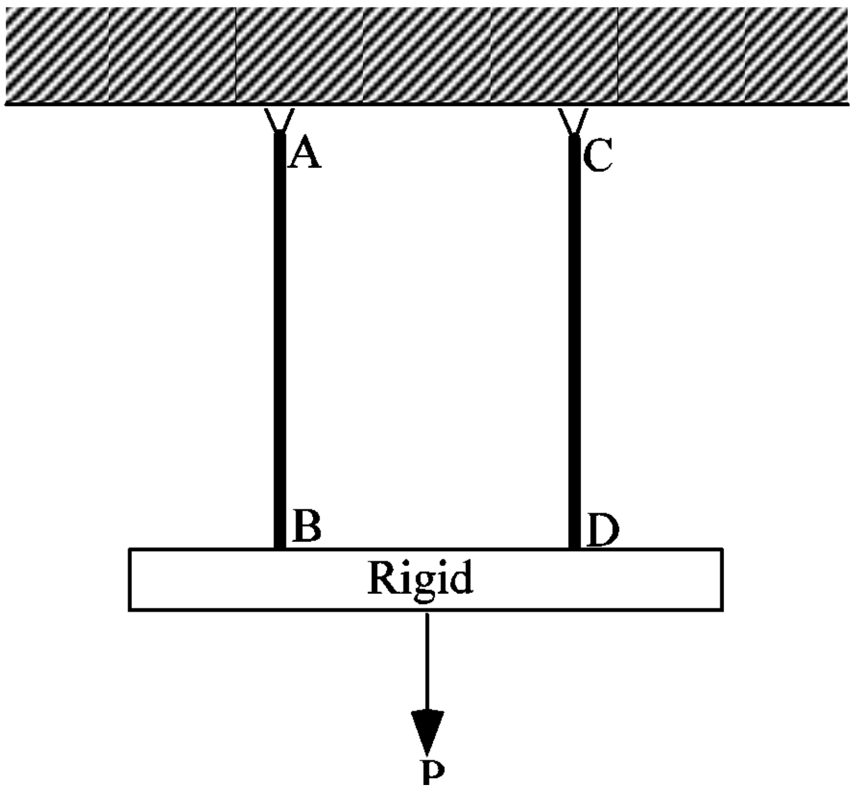
all elements are uncorrelated (ρ_{ij} = 0).

all elements are fully correlated (ρ_{ij} = 1).

PARALLEL SYSTEMS with Positive Correlation

Example:

Let the reliability indices for bars AB and CD be $\beta_{AB} = 3.00$ and $\beta_{CD} = 3.25$. The system fails (i.e., cannot carry any additional load) if both members yield



PARALLEL SYSTEMS with Positive Correlation-example

The upper bound is based on the smallest failure probability among all the elements.

$$\min_{i}\left\{P\left[F_i = 1\right]\right\} = P\,(\,F_{CD} = 1\,) = \Phi\,(\,-3.25\,) = 0.000577$$

The lower bound is based on the product of the failure probabilities, or

$$\prod_{i=1}^{n} P\left[F_i = 1\right] = \Phi(-3.00)\Phi(-3.25) = (0.00135)(0.000577) = 7.79 \text{ x } 10^{-7}$$

The exact probability of failure must be between these bounds:

$$7.79 \text{ x } 10^{-7} \leq P_f \leq 5.77 \text{ x } 10^{-4}$$

Ditlevsen Bounds for a Series System

The Ditlevsen upper bound is

$$P_F \leq \sum_{i=1}^{n} P\left(F_i = 1\right) - \sum_{\substack{i=2 \\ j<i}}^{n} \max P\left[\left(F_i = 1\right) \cap \left(F_j = 1\right)\right]$$

The Ditlevsen lower bound is

$$P_F \geq P\left(F_1 = 1\right) + \sum_{i=2}^{n} \max \left\langle P\left(F_i = 1\right) - \sum_{j=1}^{i-1} P\left[\left(F_i = 1\right) \cap \left(F_j = 1\right)\right], 0 \right\rangle$$

SYSTEMS WITH EQUALLY CORRELATED ELEMENTS

Consider a system with n elements. The strength of the i^{th} element ($i = 1, \ldots, n$) will be denoted by R_i. To be able to calculate the failure probability exactly, we must make the following assumptions:

- The strengths of the elements are all normally distributed.
- The strengths are all equally correlated, and the correlation coefficient is ρ.
- All applied loads are deterministic and constant in time.
- All elements are designed so that they have the same reliability index β_e .

SERIES SYSTEMS with Equally Correlated Elements

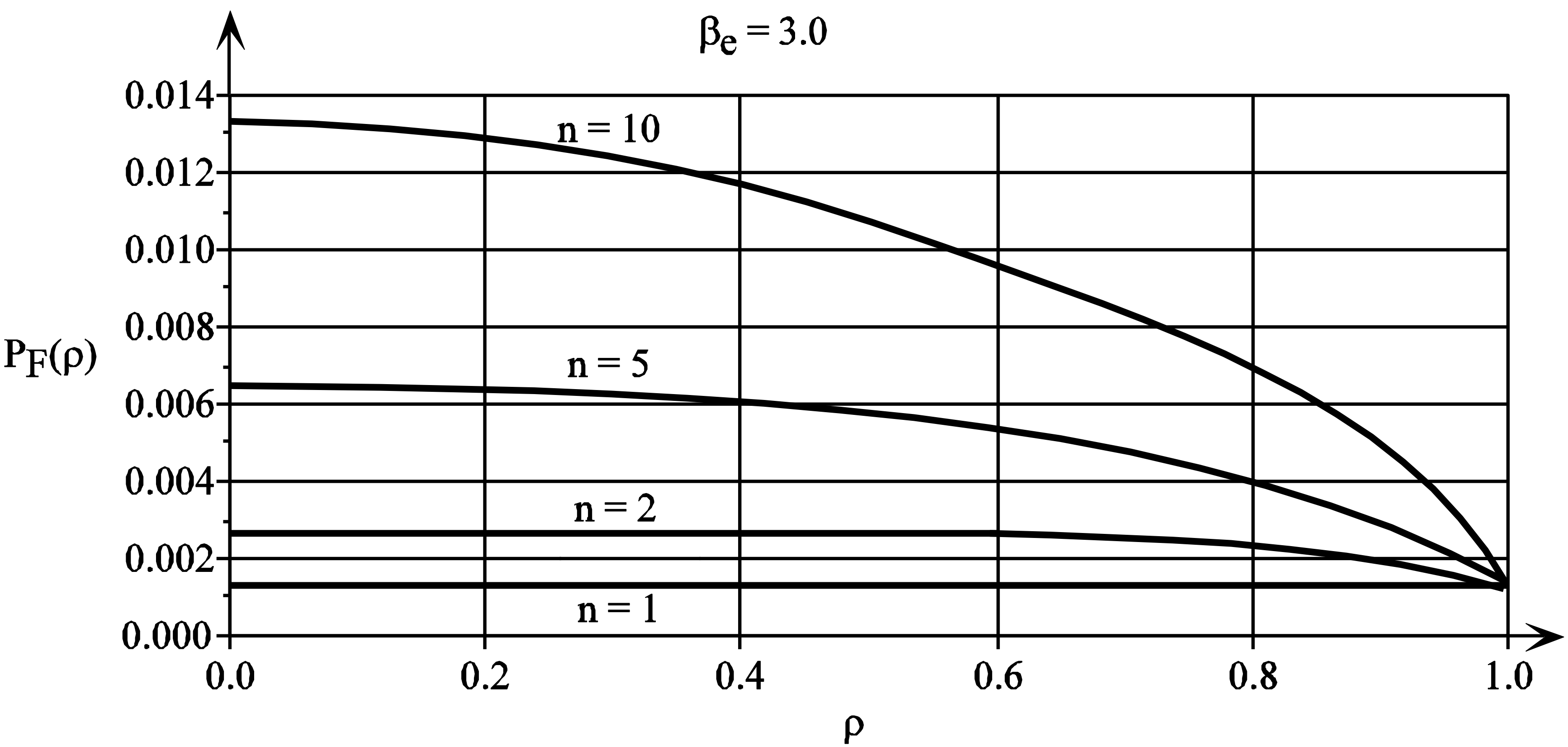
For a series system composed of n elements, Stuart (1958) derived the following formula for the probability of failure:

$$P_f = 1 - \int_{-\infty}^{\infty} \left\{ \Phi\left(\frac{\beta_e + t\sqrt{\rho}}{\sqrt{1-\rho}} \right) \right\}^n \phi(t)\, dt$$

where β_e is the reliability index for each element, $\Phi()$ and $\varphi()$ are the standard normal CDF and PDF, respectively, and ρ is the correlation coefficient which is common among all pairs of elements.

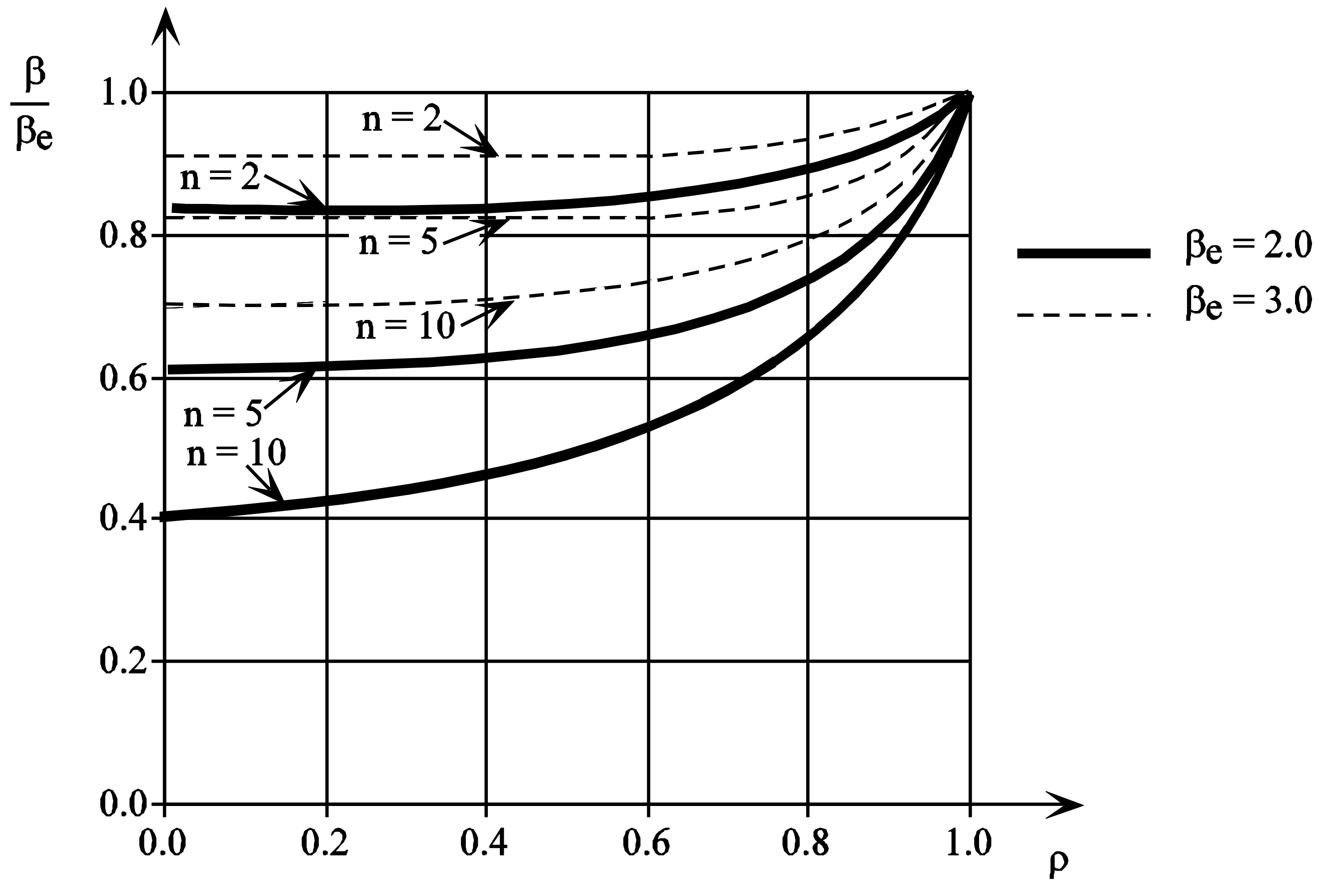
Series Systems with Equally Correlated Elements

Probability of failure for series systems with equally correlated elements

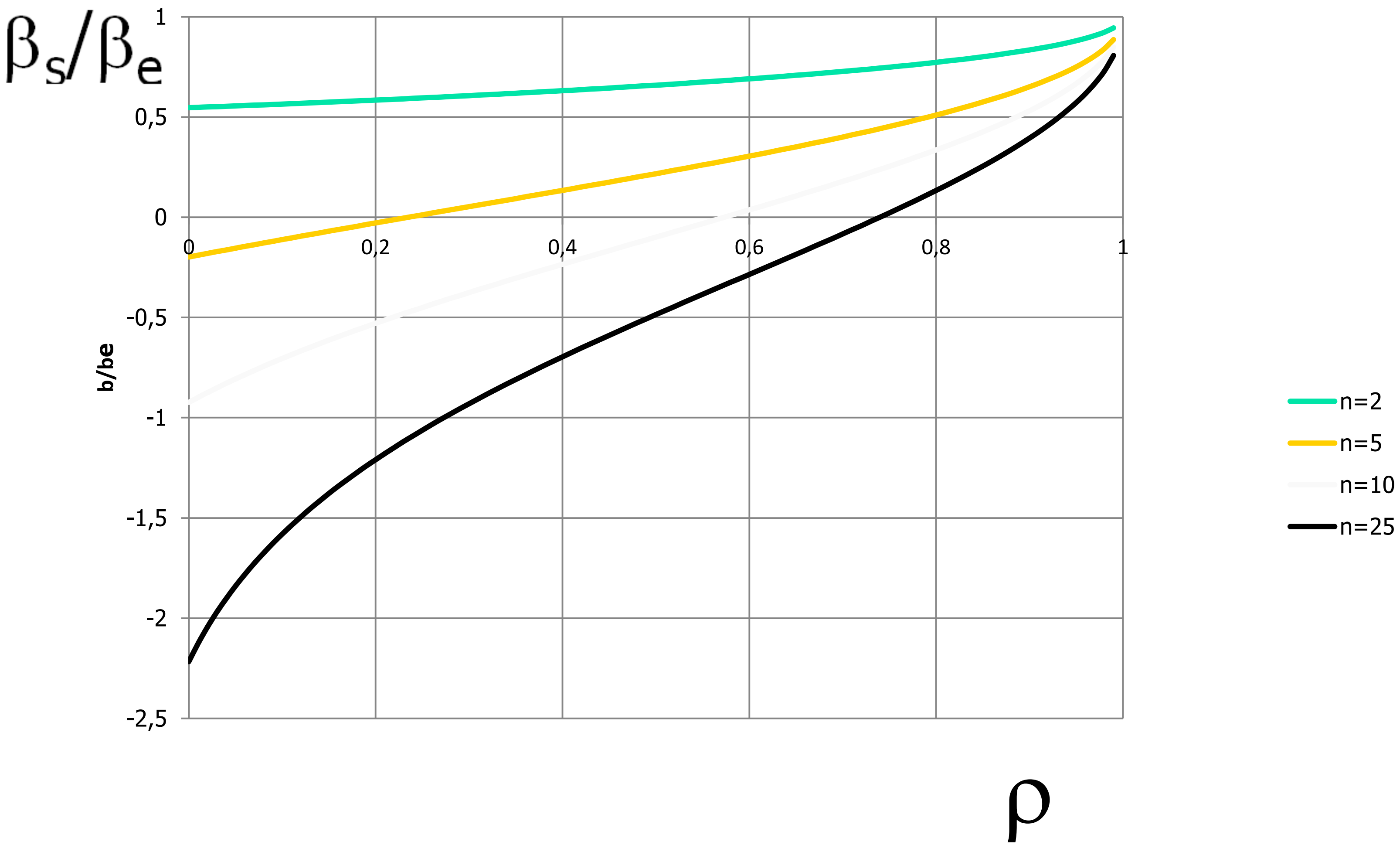


Series Systems with Equally Correlated Elements

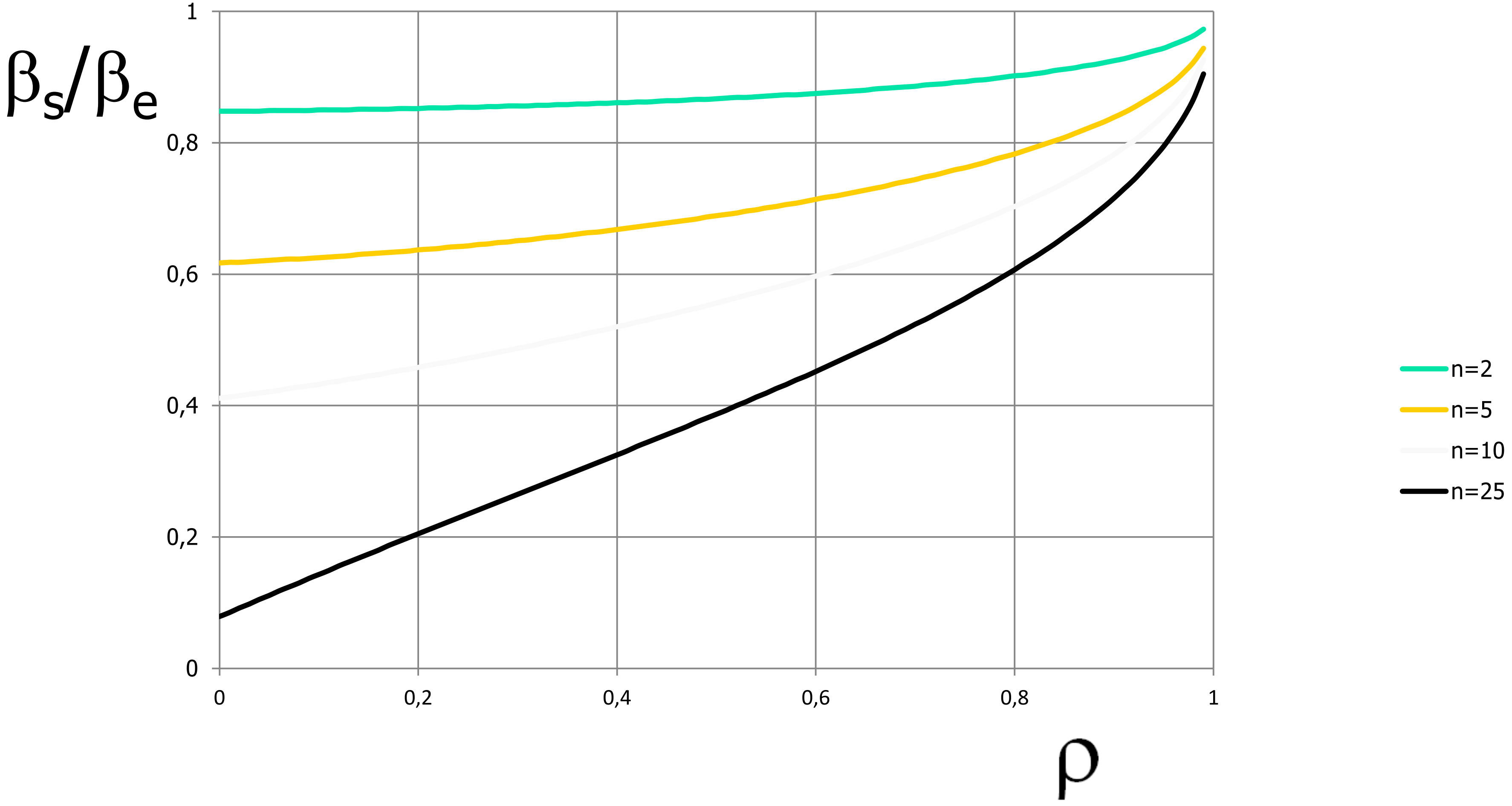
Reliability indices for a series system with equally correlated elements.



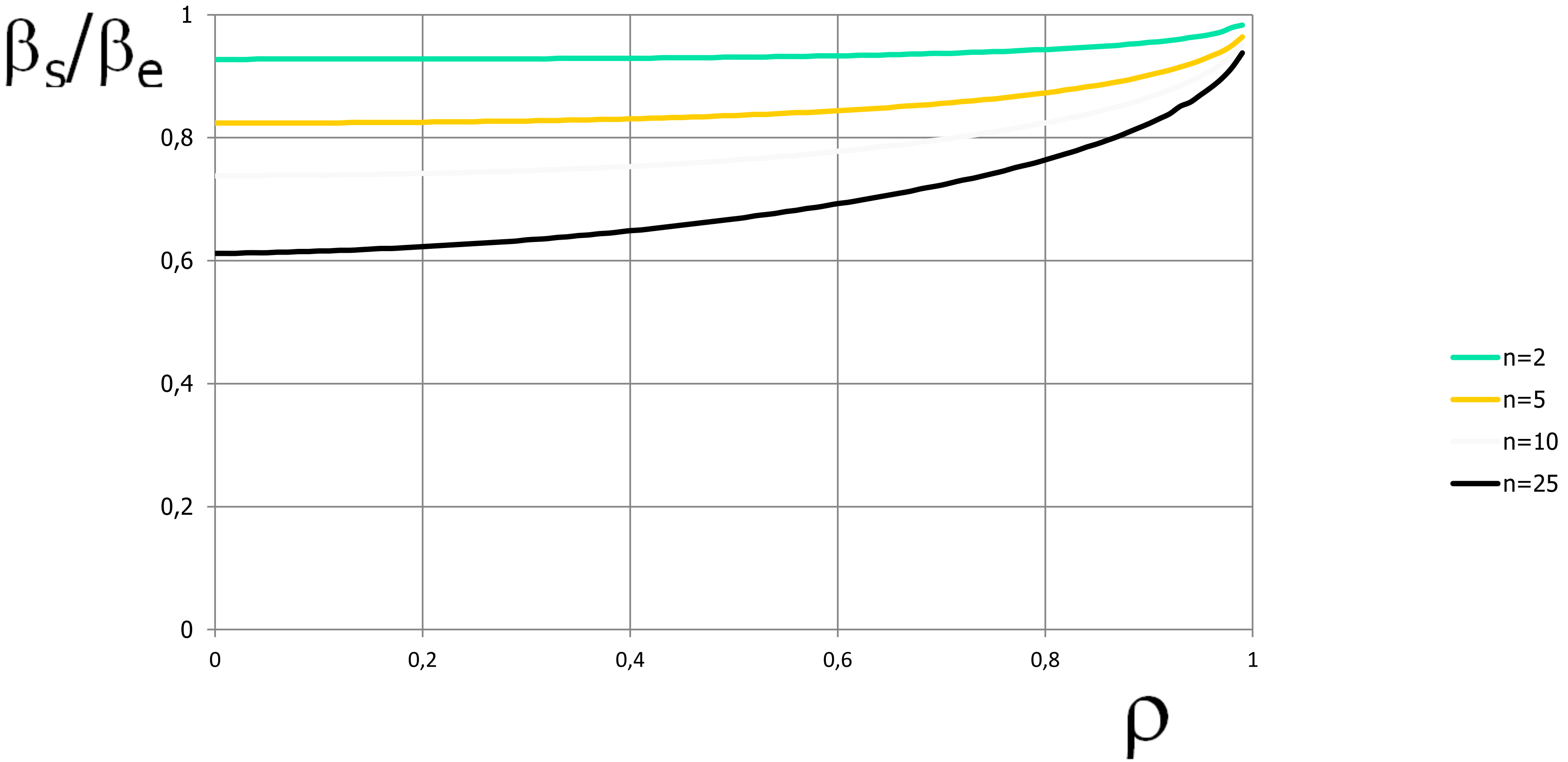
Series Systems with Equally Correlated Elements, $\beta_e = 1$



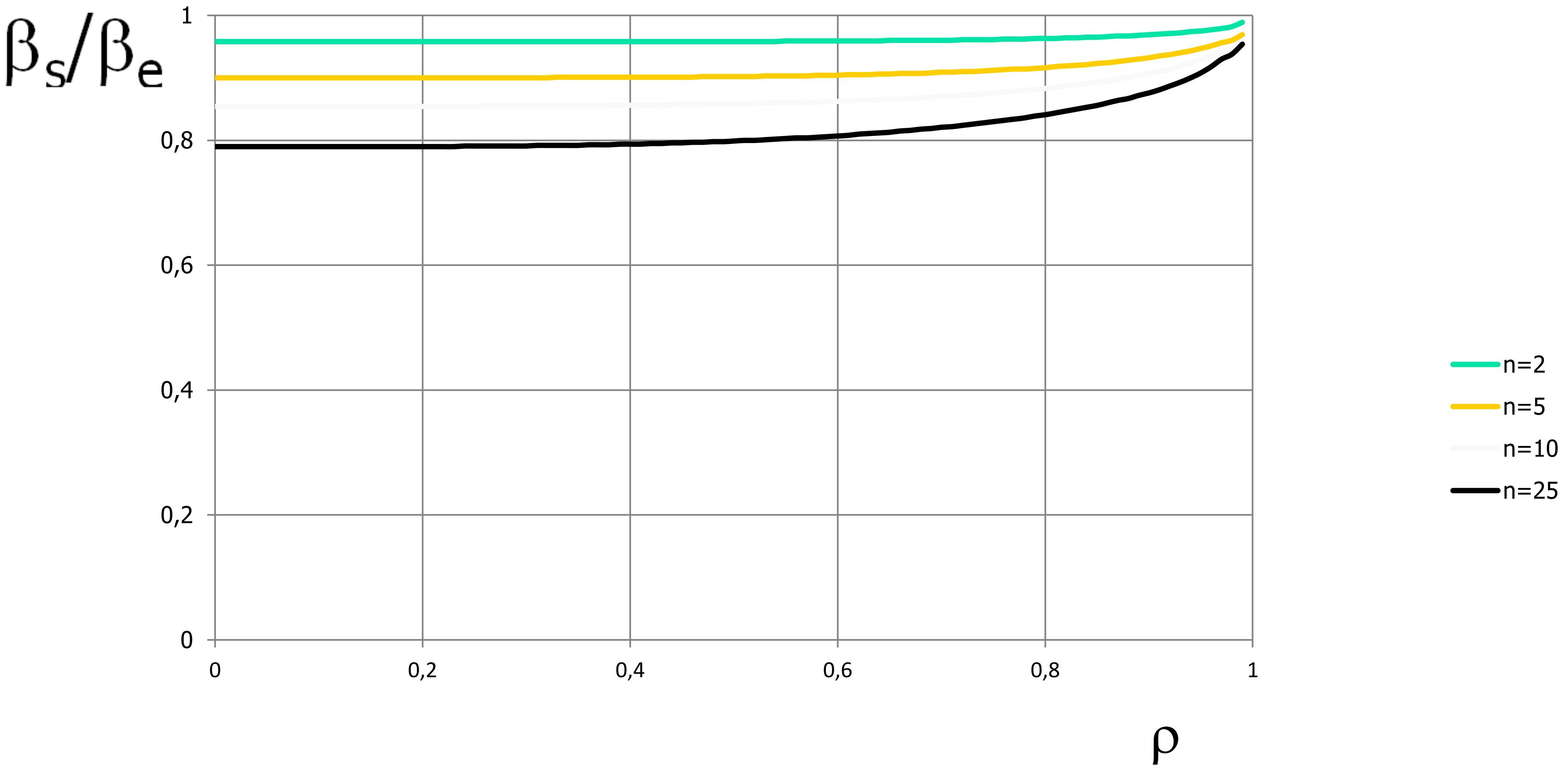
Series Systems with Equally Correlated Elements, $\beta_e = 2$



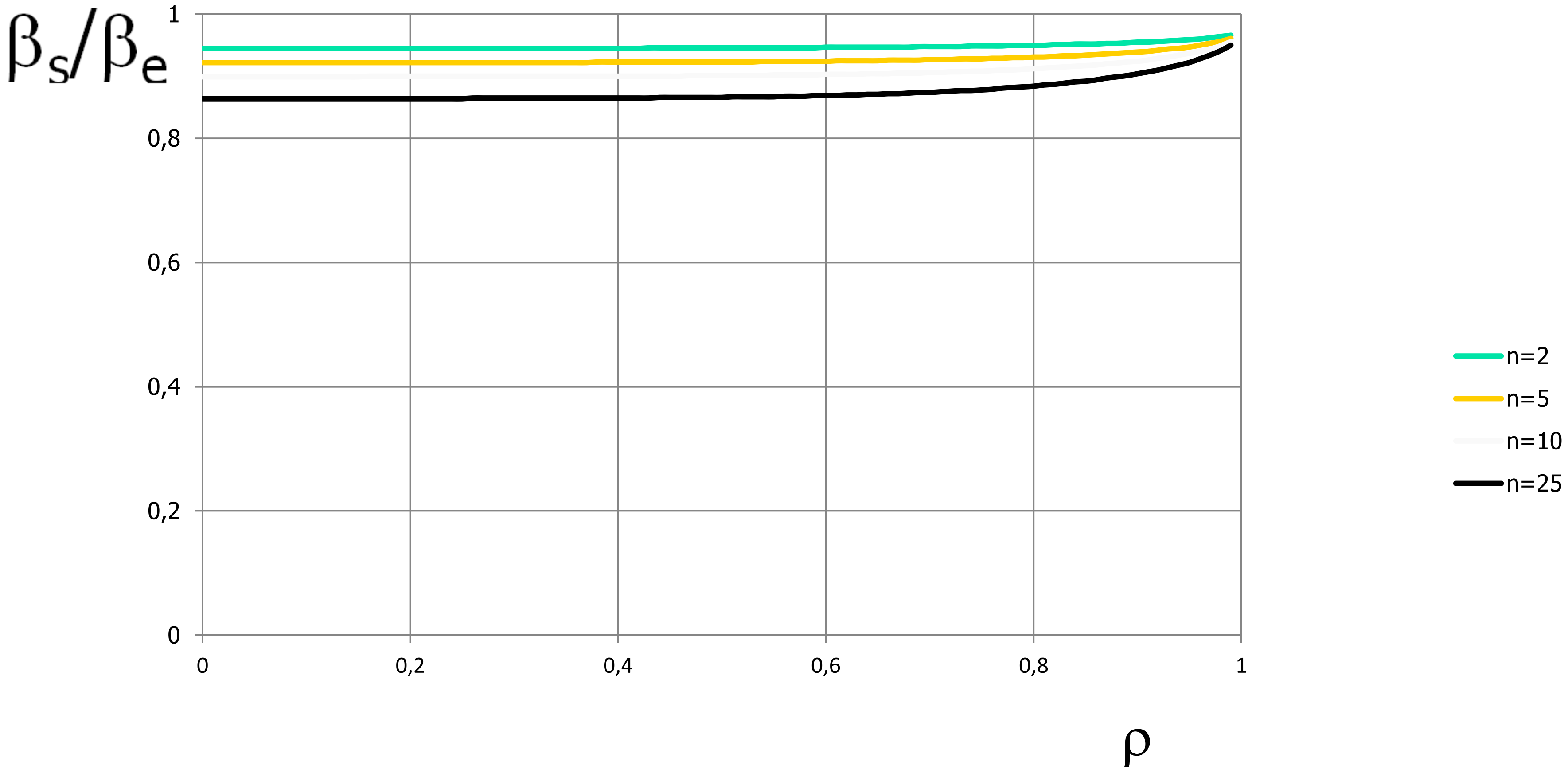
Series Systems with Equally Correlated Elements, $\beta_e = 3$



Series Systems with Equally Correlated Elements, $\beta_e = 4$



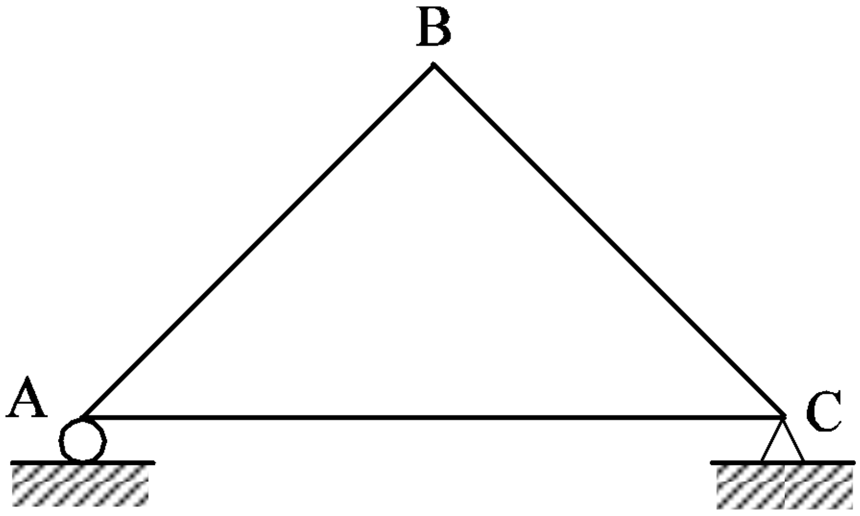
Series Systems with Equally Correlated Elements, $\beta_e = 5$



Series Systems with Equally Correlated Elements

Example:

Calculate the probability of failure for the truss. Assume β_e = 3.0 for all elements and ρ = 0.85 between elements.



Series Systems with Equally Correlated Elements-example

The truss is statically determinate. Therefore, failure of one member will lead to failure of the structure. Therefore,

$$P_f = 1 - \int_{-\infty}^{\infty} \left\{ \Phi\left(\frac{\beta_e + t\sqrt{\rho}}{\sqrt{1-\rho}} \right) \right\}^3 \phi(t)\, dt$$

In that connection: $\underline{P_f = 29.05\text{x}10^{-4}}$

Probability of failure for each element:

$$\left(p_f\right)_{element} = \Phi(-\beta_e) = \Phi(-3) = 13.50\text{x}10^{-4}$$

Parallel Systems with Equally Correlated Ductile Elements

The resistance of a parallel system with n ductile elements is:

$$R = \sum_{i=1}^{n} R_i$$

If we assume that all of the element resistances follow the same CDF:

$$\mu_R = \sum_{i=1}^{n} \mu_{R_i} = n\mu_e$$

and

$$\sigma_R^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij}\sigma_{R_i}\sigma_{R_j}$$

$$= \sum_{i=1}^{n}\sigma_e^2 + \sum_{i \neq j}\sum^{n}\sum^{n} \rho\sigma_e^2$$

$$= n\sigma_e^2 + \rho n(n-1)\sigma_e^2$$

$$= n\sigma_e^2\left(1 - \rho + n\rho\right)$$

Parallel Systems with Equally Correlated Ductile Elements

To determine the reliability index for the entire system, we must first look at how β_e is related to the mean and standard deviation of the strength for each element. For the i^{th} element, the limit state equation is

$$g(R_i) = R_i - q_i$$

The reliability index for the element is

$$\boldsymbol{\beta_e = \frac{\mu_e - q_i}{\sigma_e}}$$

and

$$q_i = \mu_e - \beta_e \sigma_e$$

Parallel Systems with Equally Correlated Ductile Elements

Since μ_e, β_e, and σ_e are the same for all elements

$$q_{tot} = n\mu_e - n\beta_e\sigma_e$$

Now, the limit state equation for the entire system is

$$g(R) = R - q_{tot}$$

and the reliability index for the system is

$$\beta_{system} = \frac{\mu_R - q_{tot}}{\sigma_R}$$

Parallel Systems with Equally Correlated Ductile Elements

The relationship between β_{system}, β_e, and ρ for a parallel system with equally correlated ductile elements:

$$\beta_{system} = \frac{n\mu_e - (n\mu_e - n\beta_e\sigma_e)}{\sqrt{n\sigma_e^2(1-\rho+n\rho)}}$$

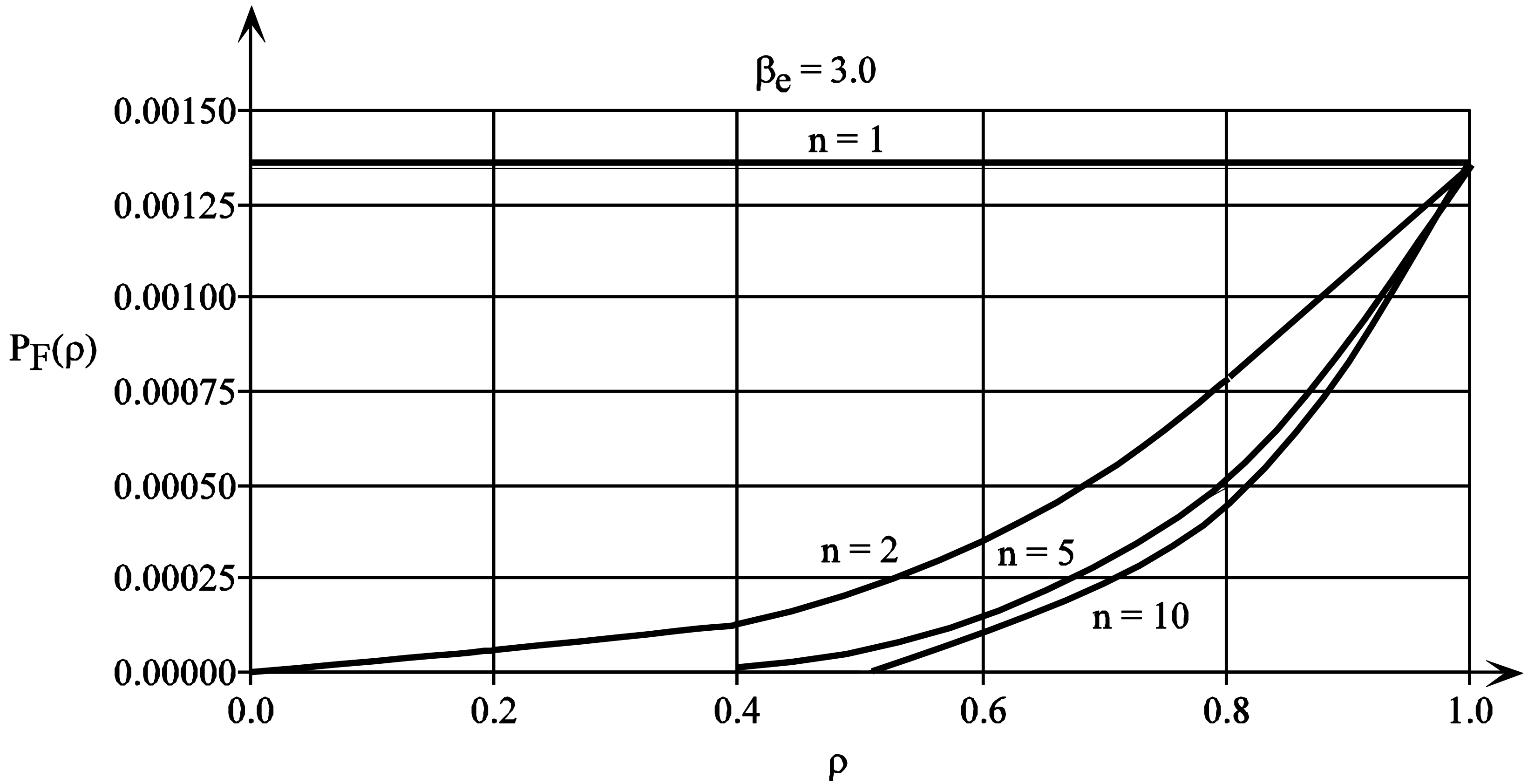
$$= \frac{n\beta_e\sigma_e}{\sigma_e\sqrt{n(1-\rho+n\rho)}}$$

$$= \beta_e\frac{\sqrt{n^2}}{\sqrt{n(1-\rho+n\rho)}}$$

$$= \beta_e\sqrt{\frac{n}{(1-\rho+n\rho)}}$$

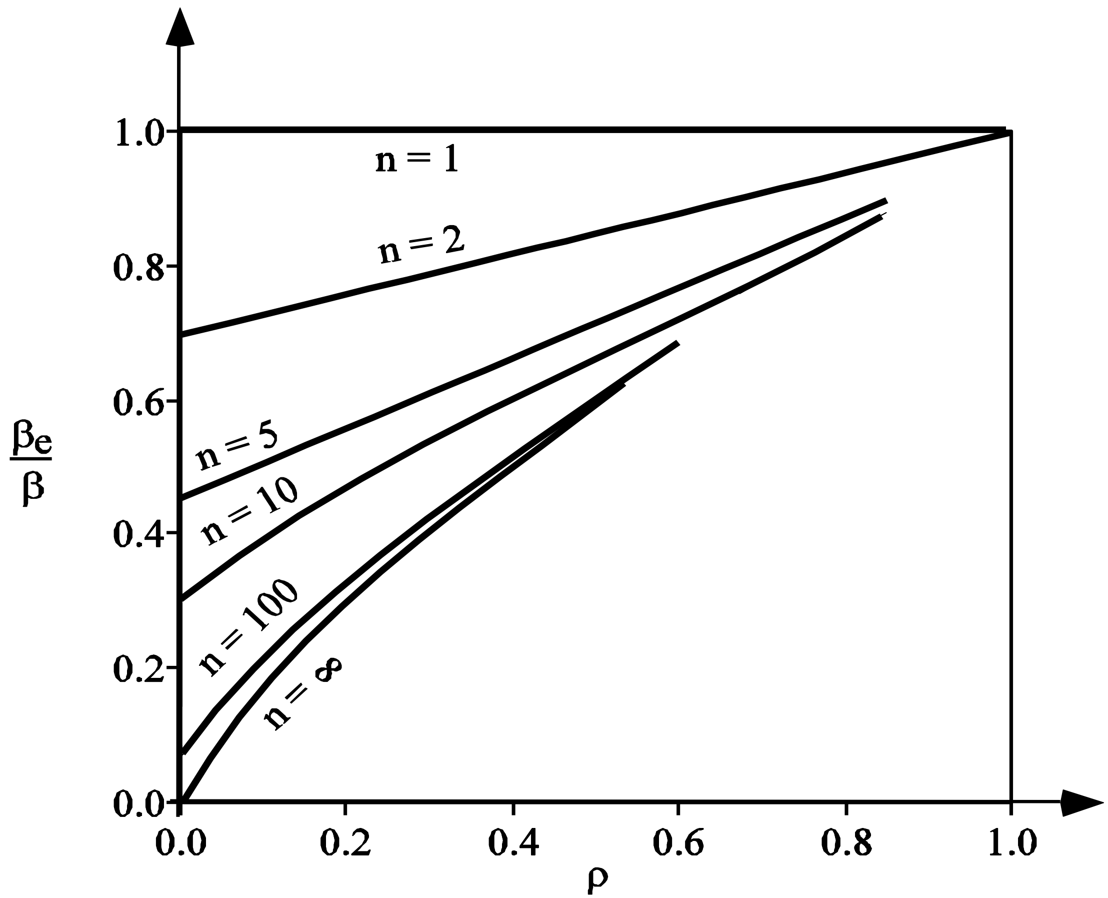
Parallel Systems with Equally Correlated Ductile Elements

Probability of failure for a parallel system with equally correlated elements.



Parallel Systems with Equally Correlated Ductile Elements

Reliability indices for parallel system with equally correlated elements.



SYSTEMS WITH UNEQUALLY CORRELATED ELEMENTS

Assumptions:

1. The strengths of the elements are all *normally distributed* with identical distribution parameters μ_e and σ_e.
2. All applied loads are deterministic and constant in time.

3. All elements are designed so that they have the same reliability index, β_e

Parallel System with Ductile Elements

The correlation matrix describing the correlations between elements is

$$[\rho] = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & & \rho_{2n} \\ \vdots & & \ddots & \\ \rho_{n1} & \rho_{n2} & \cdots & 1 \end{bmatrix}$$

The reliability index for the system, β_{system}, is

$$\beta_{system} = \frac{\mu_R - q_{tot}}{\sigma_R}$$

Parallel System with Ductile Elements

where $\mu_R = n\mu_e$

and
$$\sigma_R^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\sigma_{R_i}\sigma_{R_j} = \sum_{i=1}^{n}\sigma_e^2 + \sum_{i\neq j}^{n}\sum^{n}\rho_{ij}\sigma_e^2$$
$$= n\sigma_e^2 + \sigma_e^2\left(\sum_{i\neq j}^{n}\sum^{n}\rho_{ij}\right) = \sigma_e^2\left[n + \sum_{i\neq j}^{n}\sum^{n}\rho_{ij}\right]$$

Substituting these results into $\beta_{system} = \dfrac{\mu_R - q_{tot}}{\sigma_R}$ and

making some rearrangements, we obtain:

Parallel System with Ductile Elements

$$\beta_{\text{system}} = \frac{n\mu_e - (n\mu_e - n\beta_e\sigma_e)}{\sqrt{\sigma_e^2\left[n + \sum_{i\neq j}^{n}\sum^{n}\rho_{ij}\right]}} = \frac{n\beta_e\sigma_e}{\sqrt{\sigma_e^2\left[n + \sum_{i\neq j}^{n}\sum^{n}\rho_{ij}\right]}}$$

$$= \beta_e\sqrt{\frac{n^2}{n + \sum_{i\neq j}^{n}\sum^{n}\rho_{ij}}} = \beta_e\sqrt{\frac{n}{1 + \frac{1}{n}\sum_{i\neq j}^{n}\sum^{n}\rho_{ij}}}$$

Parallel System with Ductile Elements

An "average correlation" coefficient, $\bar{\rho}$, is defined as

$$\bar{\rho} = \frac{1}{n(n-1)} \sum_{i \neq j}^{n} \sum^{n} \rho_{ij}$$

Finally

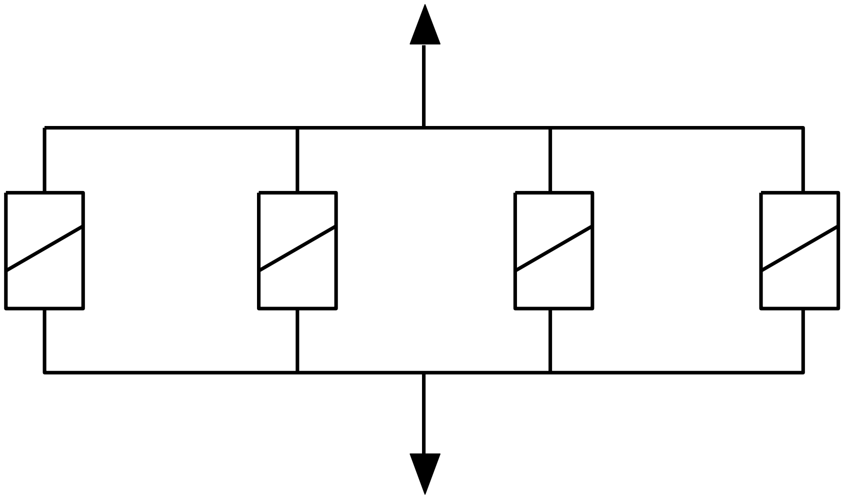
$$\beta_{system} = \beta_e \sqrt{\frac{n}{1 + (n-1)\bar{\rho}}}$$

Parallel System with Ductile Elements

Example

The element reliability index for all elements is β_e = 4.0.
The correlation matrix is given as

$$[\rho]=\begin{bmatrix} 1 & 0.4 & 0.2 & 0.1 \\ 0.4 & 1 & 0.2 & 0.3 \\ 0.2 & 0.2 & 1 & 0.5 \\ 0.1 & 0.3 & 0.5 & 1 \end{bmatrix}$$



Parallel System with Ductile Elements-example

Solution:

$$\bar{\rho} = \frac{1}{n(n-1)} \sum_{i \neq j}^{n} \sum^{n} \rho_{ij}$$

$$= \frac{1}{4(3)} [0.4 + 0.2 + 0.1 + 0.4 + 0.2 + 0.3 + 0.2 + 0.2 + 0.5 + 0.1 + 0.3 + 0.5]$$

$$= \frac{3.4}{12} = 0.283$$

$$\beta_{\text{system}} = \beta_e \sqrt{\frac{n}{1 + (n-1)\bar{\rho}}} = 4.0 \sqrt{\frac{4}{1 + (3)(0.283)}} = 5.88$$

Parallel System with Ductile Elements-example

If all elements are uncorrelated, then $\bar{\rho} = 0$ and

$$\beta_{\text{system}} = \beta_e \sqrt{\frac{n}{1+(n-1)\bar{\rho}}} = 4.0\sqrt{\frac{4}{1+(3)(0)}} = 8.0$$

If all elements are perfectly correlated, then $\bar{\rho} = 1$ and

$$\beta_{\text{system}} = \beta_e \sqrt{\frac{n}{1+(n-1)\bar{\rho}}} = 4.0\sqrt{\frac{4}{1+(3)(1)}} = 4.0$$

Series System with Correlated Elements

Probability of failure can be approximated by

$$P_f = 1 - \int_{-\infty}^{\infty} \left\{ \Phi\left(\frac{\beta_e + t\sqrt{\bar{\rho}}}{\sqrt{1-\bar{\rho}}} \right) \right\}^n \phi(t)\, dt$$

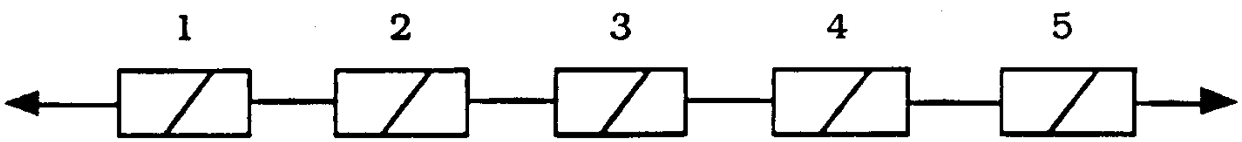
This approximation is typically very good for small values of n (number of elements). The approximation can be improved for larger n by using the following formula:

$$P_f = P_f(n, \bar{\rho}) - \left[P_f(2, \bar{\rho}) - P_f(2, \rho_{max}) \right]$$

Series System

Example

Calculate the probability of failure for the series system.



The reliability index of each element is $\beta_e = 3.00$. The correlation matrix is

$$\overline{\overline{C}} = \begin{bmatrix} 1 & 0.5 & 0.2 & 0.1 & 0 \\ 0.5 & 1 & 0.5 & 0.2 & 0.1 \\ 0.2 & 0.5 & 1 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.5 & 1 & 0.5 \\ 0 & 0.1 & 0.2 & 0.5 & 1 \end{bmatrix}$$

Series System-example

The upper and lower bounds for the probability of failure can be calculated by using Ditlevsen formulas.

$$63.8\,10^{-4} \leq P_F \leq 64.2\,10^{-4}$$

On the other hand, P_f can also be determined by

$$\bar{\rho} = \frac{1}{n(n-1)} \sum^{n} \sum_{i \neq j}^{n} \rho_{ij} = \ldots = \frac{3.4}{12} = 0.283$$

So, $P_F = 65.5\,10^{-4}$ $\left(\text{for } \rho = \bar{\rho},\ n = 5\right)$

Series System-example

or more accurately

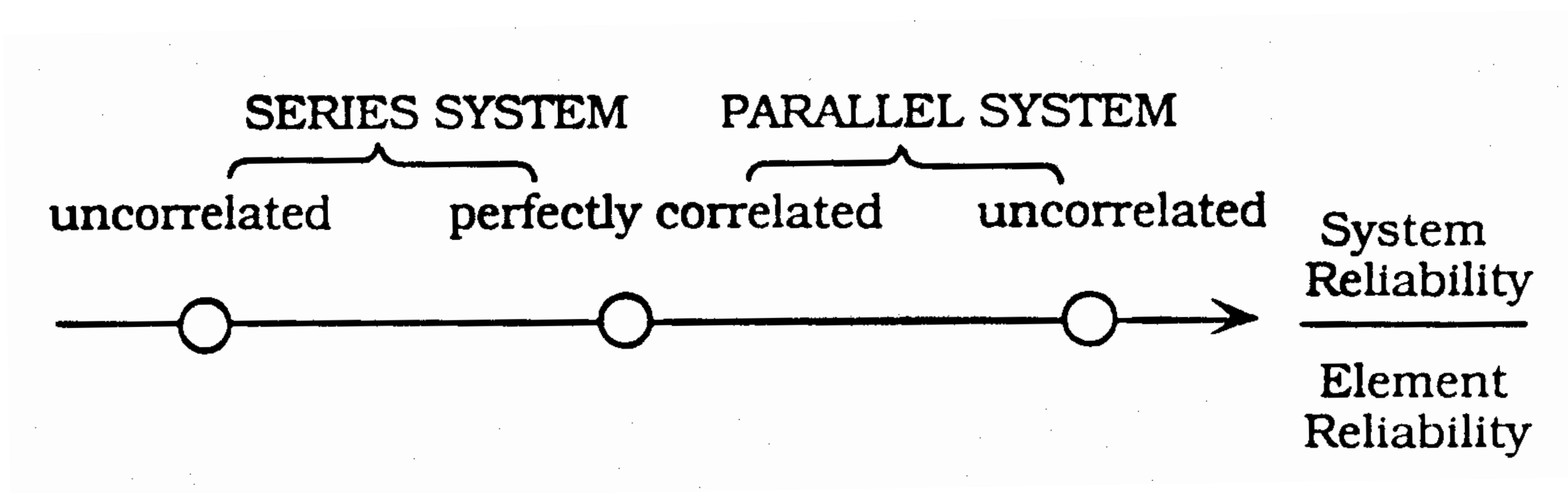
$$P_F = P_F(0.28) - (P_{F_2}(0.28) - P_{F_2}(0.50)) = 65.5\,10^{-4} - (26.8\,10^{-4} - 26.2\,10^{-4})$$

$n = 2$ ρ_{max}

$$\boxed{P_F = 64.9\,10^{-4}}$$

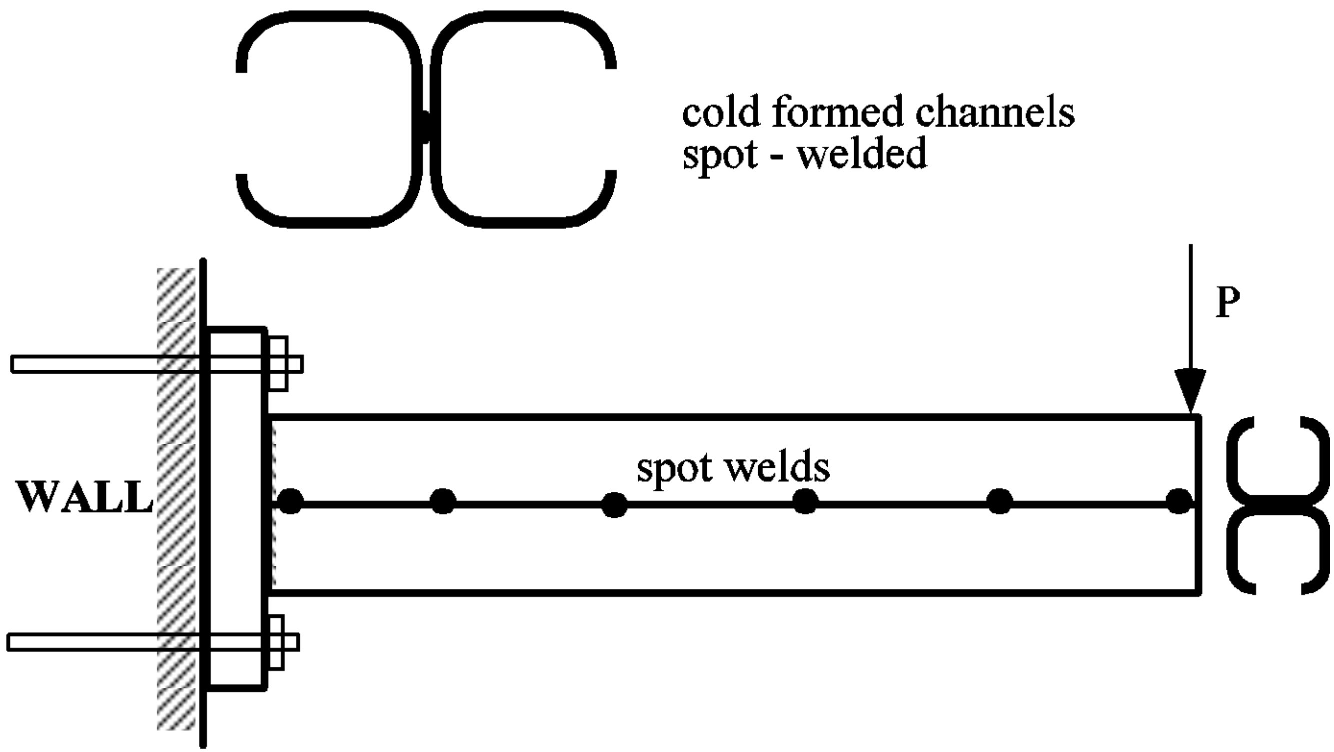
SUMMARY

System	Correlation	Description
Series	$\rho=0$	System reliability < Element reliability
	$\rho=1$	System reliability = Element reliability
Parallel	$\rho=0$	System reliability > Element reliability
	$\rho=1$	System reliability = Element reliability



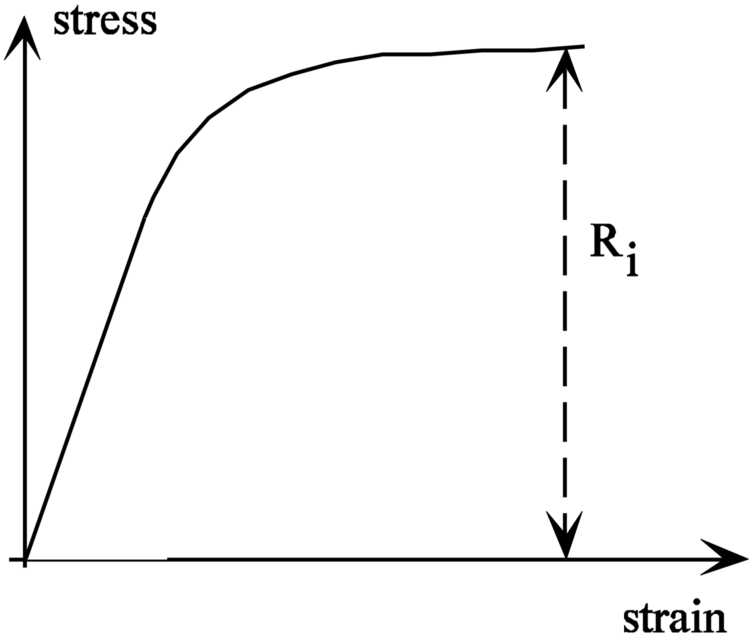
EXAMPLE

Consider spot welds in cold-formed channels. The shear capacity of the spot welds determines the capacity of the cantilever. The spot welds can be considered as a parallel system, since all of them must fail before the system fails.



EXAMPLE

The shear force per weld is denoted by V_i and is a constant. The weld strength for each weld is R_i. The shear strength is determined by a stress-strain curve



Two extreme cases can be considered:

1. Perfectly correlated welds (if one is bad, all others are equally bad).
2. Uncorrelated welds (if one is bad, the others may be bad, good, or average).

EXAMPLE

The total shear capacity

$$R_V = R_1 + R_2 + \ldots + R_n + \sum_{i=1}^{n} R_i$$

$$\overline{R}_V = \sum_{i=1}^{n} \overline{R}_i \qquad \text{(whether correlated or not)}$$

σ_{R_V} depends whether correlated or not.

$$\sigma^2_{R_V} = \sigma^2_{R_1} + \sigma^2_{R_2} + \ldots + \sigma^2_{R_n} + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \underbrace{\mathrm{Cov}\,(R_i, R_j)}$$

for uncorrelated R_i's
this is zero.

EXAMPLE

Derivation of the above formula:

let $\quad Y = \sum_{i=1}^{n} a_i X_i,\quad$ where X_i are random variables

(correlated or uncorrelated)

then

$$\bar{Y} = E\,(Y) = \sum_{i=1}^{n} a_i \bar{X}_i$$

$$\sigma_Y^2 = E\,(Y - \bar{Y})^2 = E\left(\sum_{i=1}^{n} a_i X_i - \sum_{i=1}^{n} a_i \bar{X}_i\right)^2 =$$

$$= E\left[\sum_{i=1}^{n} a_i \left(X_i - \bar{X}_i\right)\right]^2 = E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \left(X_i - \bar{X}_i\right)\left(X_j - \bar{X}_j\right)\right] =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j\, E\left[\left(X_i - \bar{X}_i\right)\left(X_j - \bar{X}_j\right)\right] =$$

$$= \sum_{i=1}^{n} a_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \,\mathrm{Cov}\left(X_i, X_j\right)$$

EXAMPLE

For uncorrelated welds,

$$\sigma^2_{R_v} = \sum_{i=1}^{n} \sigma^2_{R_i} \quad \rightarrow \quad V_{R_v} = \frac{\sqrt{\sum_{i=1}^{n} \sigma^2_{R_i}}}{\overline{R}_v}$$

if R_i's are random variables with $F_{R_i}(x) = F_1(x)$, (all have the same distribution function), then

$$\overline{R}_v = n\,\overline{R}_1 \; ; \quad \sigma^2_{R_v} = n\,\sigma^2_{R_1}$$

$$V_{R_v} = \frac{\sqrt{n\,\sigma^2_{R_1}}}{n\,\overline{R}_1} = \left(\frac{\sigma_{R_1}}{\overline{R}_1}\right)\left(\frac{\sqrt{n}}{n}\right) = \frac{1}{\sqrt{n}} V_{R_1}$$

$$\boxed{V_{R_v} = \frac{1}{\sqrt{n}} V_{R_1}}$$

For uncorrelated elements the coefficient of variation for the system is $\frac{1}{\sqrt{n}}$ that for one element.

EXAMPLE

For perfectly correlated welds,

$$\rho_{R_iR_j} = \frac{\mathrm{Cov}\,(R_iR_j)}{\sigma_{R_i}\,\sigma_{R_j}} = 1 \quad \text{for all } i \neq j$$

so $\quad \mathrm{Cov}\,(R_i,\, R_i) = \sigma_{R_i}\,\sigma_{R_j}$

$$\sigma^2_{R_v} = \sum_{i=1}^{n} \sigma^2_{R_i} + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sigma_{R_i}\,\sigma_{R_j}$$

If all distributions are the same, then

$$\sigma^2_{R_v} = n\;\sigma^2_{R_1} + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sigma^2_{R_1} = \sigma^2_{R_1}\left[n + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 1\right]$$

but

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 1 = (n-1) + (n-2) + \ldots + 2 + 1 = \frac{(n-1)(n)}{2}$$

EXAMPLE

so

$$\sigma^2_{R_v} = \sigma^2_{R_1}\left[n + 2\,\frac{(n-1)(n)}{2}\right] = n^2\,\sigma^2_{R_1}$$

Therefore,

$$V_{R_v} = \frac{\sqrt{n^2\,\sigma^2_{R_1}}}{n\,\bar{R}_1} = V_{R_1}$$

$$\boxed{V_{R_v} = V_{R_1}}$$

For correlated elements the coefficient of variation for the system is = that for one element.

Therefore, for partially correlated elements,

$$\boxed{\frac{1}{\sqrt{n}}V_{R_1} \le V_{R_v} \le V_{R_1}}$$