Complex Dynamics of Pendula for Energy *Extraction and Harvesting*

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- Polytechnic University of Marche: Prof S Lenci
- University of Kyoto: Dr Y Yokoi
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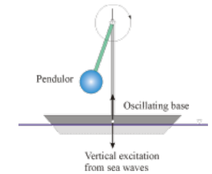
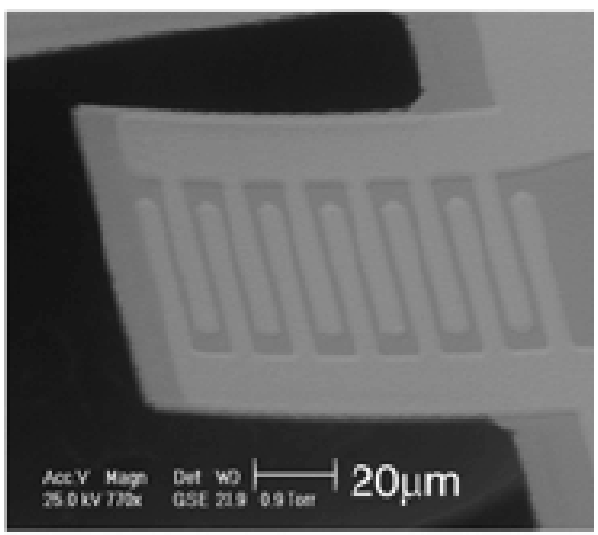
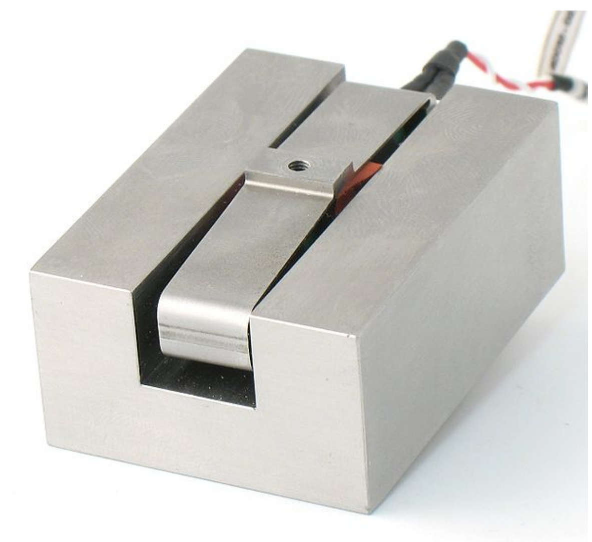
- Indian Institute of Technology Kharagpur: Prof A Chaterjee
- Lodz University of Technology: Prof T Kapitaniak and Dr J Wojewoda
- Indian University of Technology Madras: Prof S Narayanan

Motivation

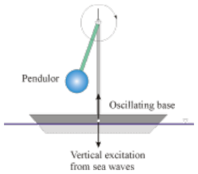


Examples of Mechanical Energy Extraction and Harvesting

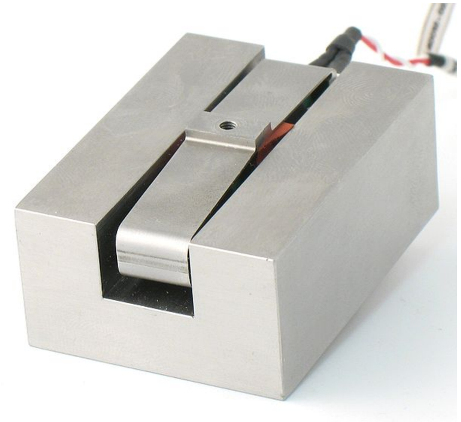




Motivation



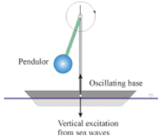
VEH-APA 400M-MD [Courtesy of CEDRAT]



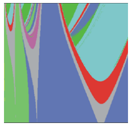
References	Unit	APA400M-MD based VEH System
Notes		
Technological baseline		*APA400M-MD*
Harvested vibration frequency	*Hz*	*110*
Harvested vibration amplitude	*µm p-p*	*45*
Max Harvesting efficiency (with Fly Back converter)	*%*	*48% (36%)*
Max Harvested Power	*mW*	*95*
Dimensions of the proof mass	*mm^3*	*50 x 32 x 22*
Weight of the proof mass	*grams*	*270*
Mechanical interface		*2 flat surfaces 5*10 mm2 with M2.5 threaded hole*
Electrical interface		*TBD*

0.35

Presentation Outline

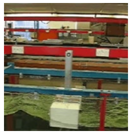


Motivation

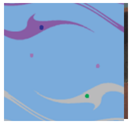


Rotary dynamics of parametric pendulum

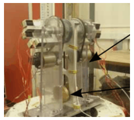
Intermediate experimental verification



Experimentation in a water flume

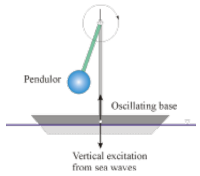


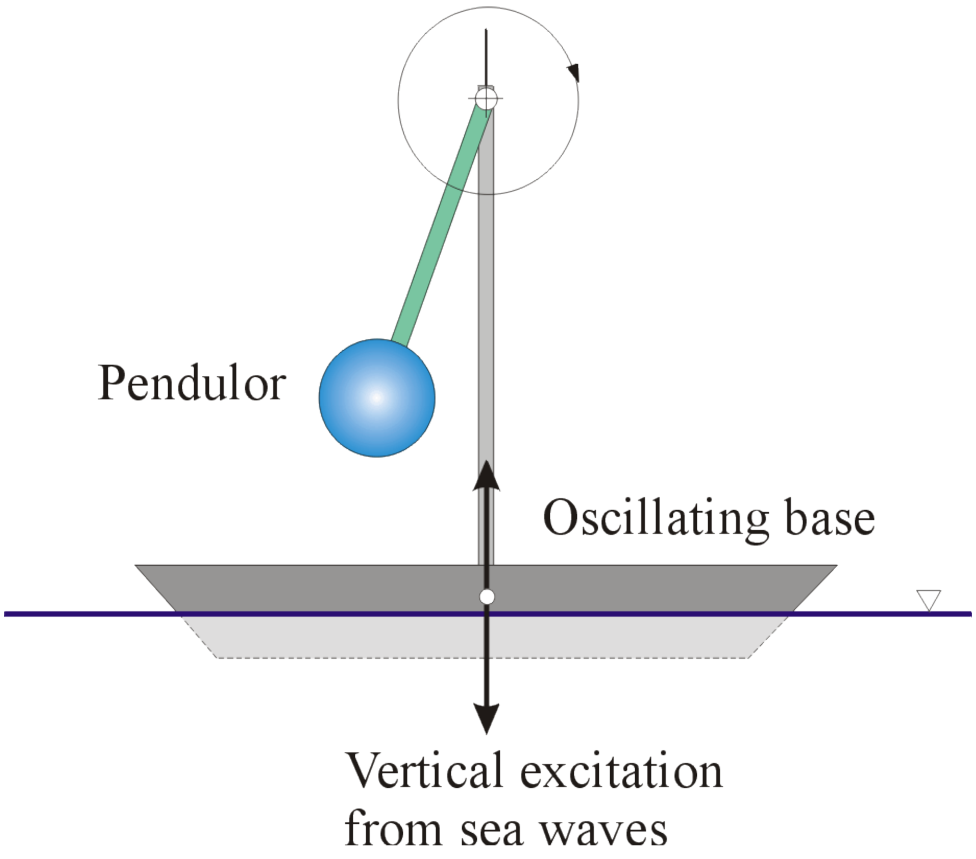
Enhacements and control



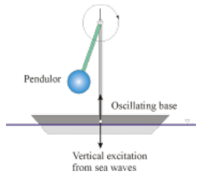
Pendula systems and stochastic wave excitation

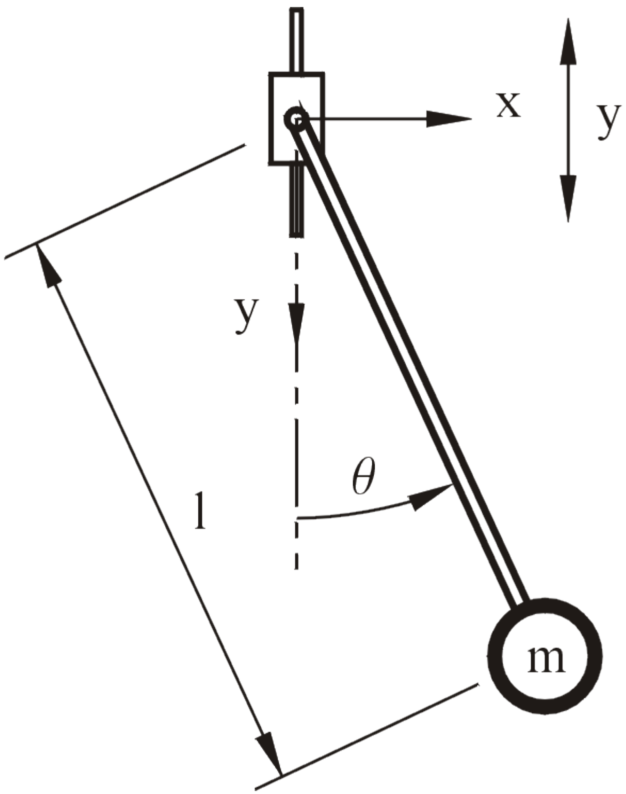
Parametric Pendulum





Parametric Pendulum





$$\ddot{\theta} + \gamma\dot{\theta} + (1 + p\cos(\omega\tau))\sin\theta = 0$$

$$\tau = \omega_n t, \omega_n = \sqrt{g/l},$$

$$\gamma = c/(m\omega_n),\ p = Y\Omega^2/g \text{ and } \omega = \Omega/\omega_n.$$

Mathieu Equation

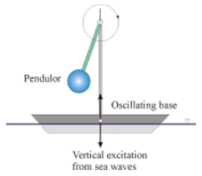
$$\ddot{\theta} + (1 + p\cos(\omega\tau))\theta = 0$$

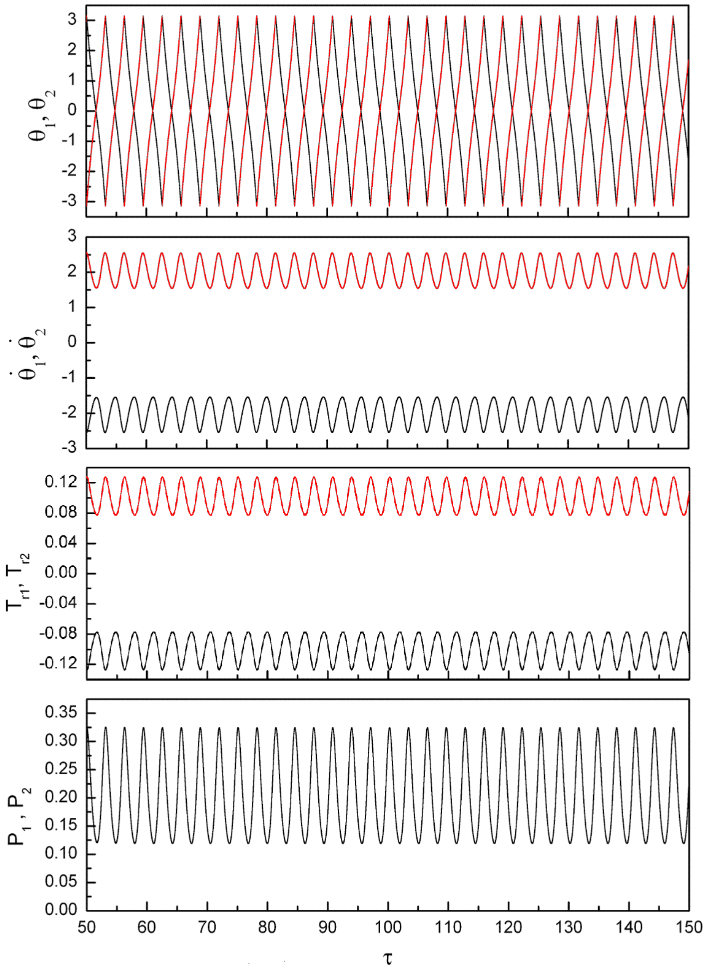
$$'' = d^2/d\bar{\tau}^2, \bar{\tau} = \omega\tau = \Omega t, \bar{\omega} = 1/\omega$$

$$\epsilon = p/\omega^2 = Y/\ell \text{ is a small number.}$$

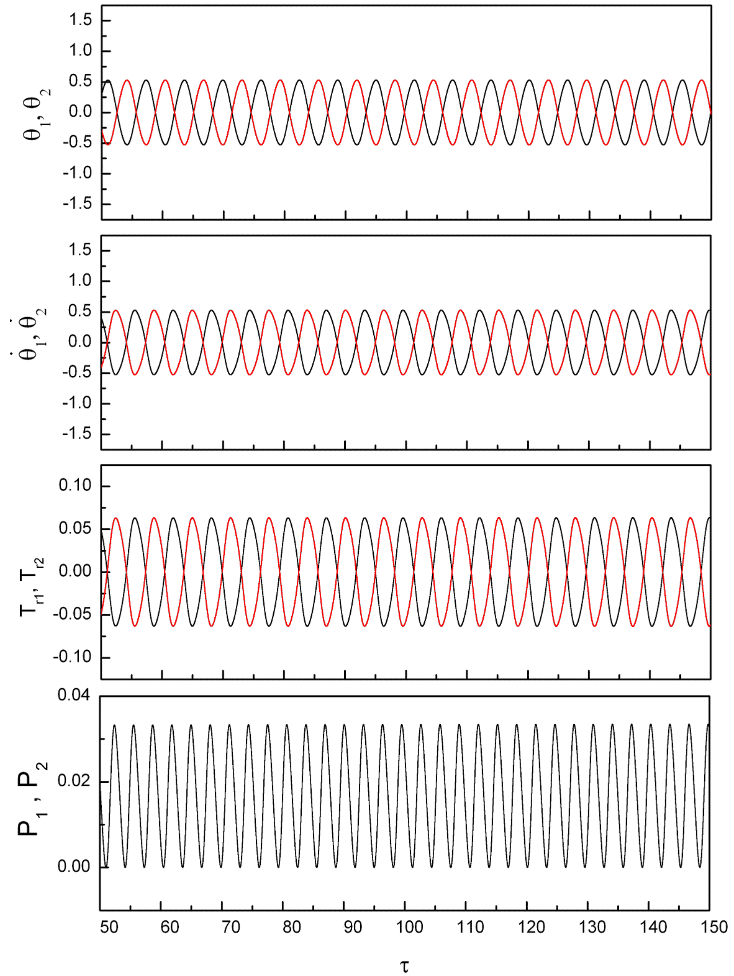
$$\theta'' + (\bar{\omega}^2 + \epsilon\cos\bar{\tau})\theta = 0$$

Parametric Pendulum



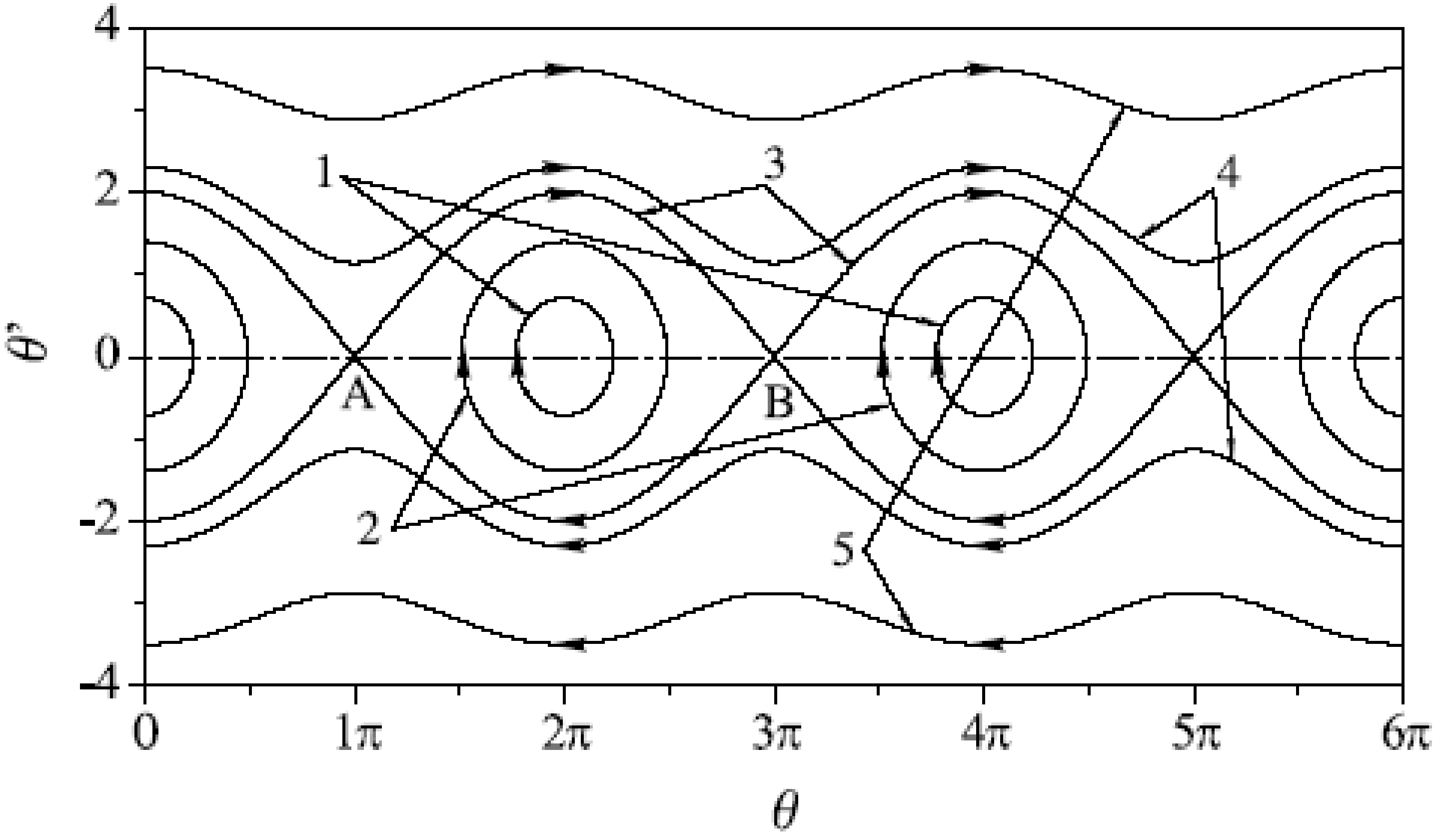


Rotations



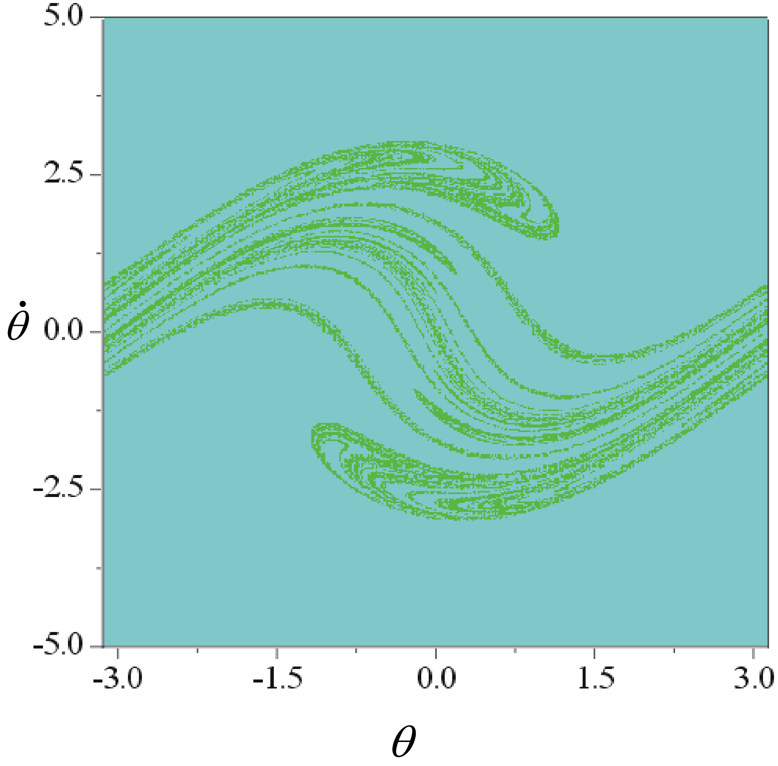
Oscillations

Parametric Pendulum



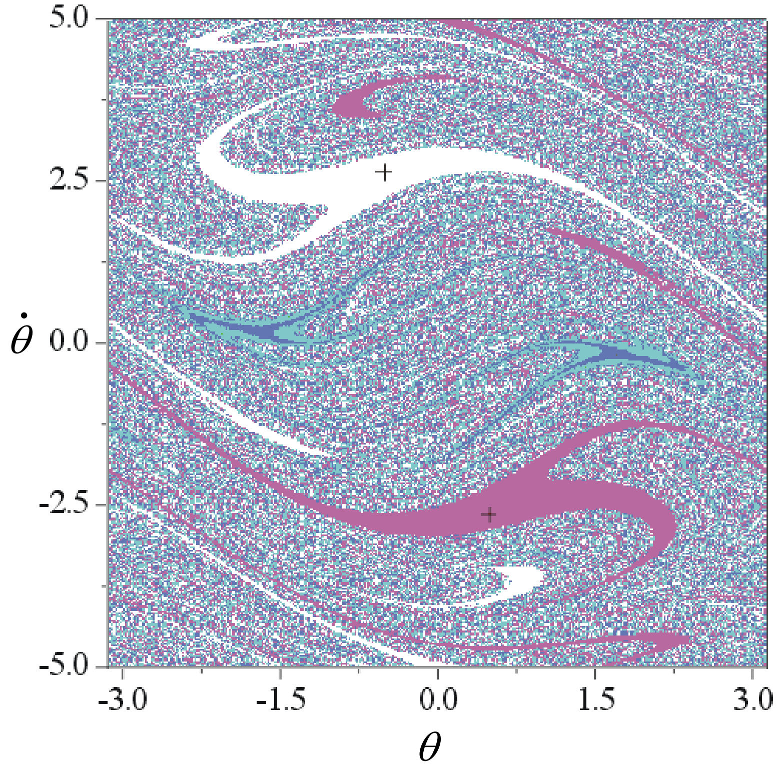
Parametric Pendulum

$p = 1, \ \omega = 1.6$



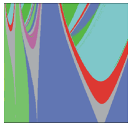
Chaotic attractor

$p = 1.1, \omega = 2$



Co-existing of four periodic attractors: two period-one rotations, a period-two and a period-six oscillations.

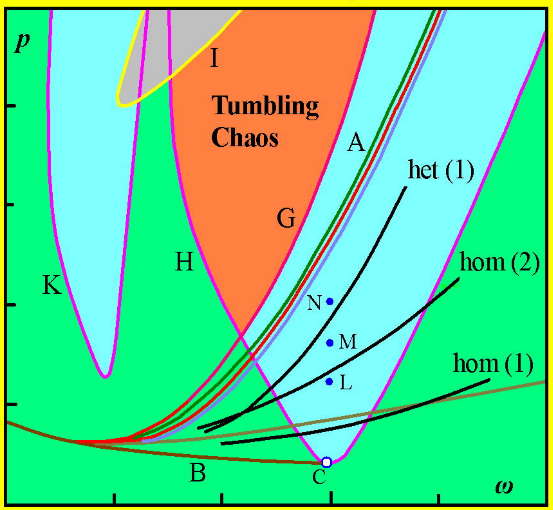
Parametric Pendulum



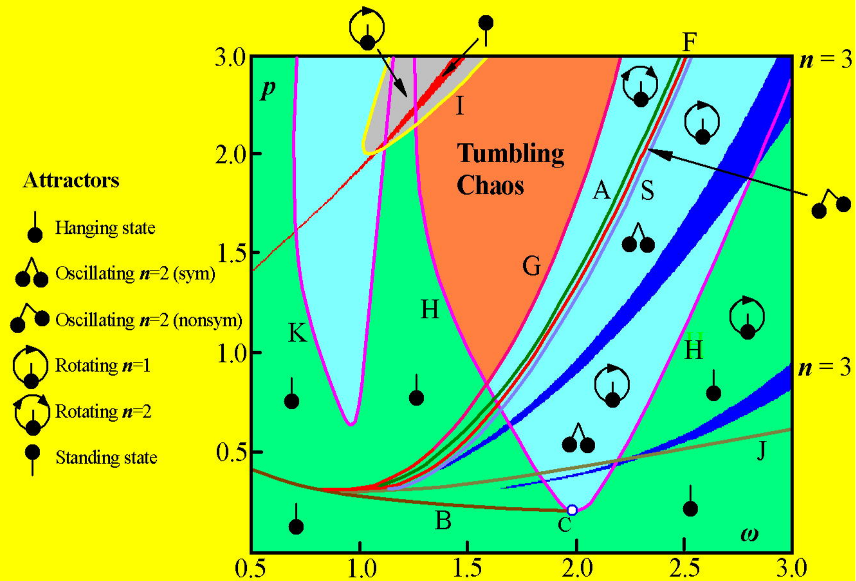
The Parametrically excited Pendulum

$$\theta'' + c\,\theta' + (1 + p \sin \omega t) \sin \theta = 0 \quad \text{with} \quad c = 0.1$$

Some Global Tangencies



hom (1) homoclinic tangle of hilltop saddle
hom (2) homoclinic tangle of rotating $n = 1$ saddles (created at J)
het (1) heteroclinic tangle of rotating $n = 1$ saddles with hanging saddle



Bifurcation Arcs

H Instability of the hanging state, sub–critical to the left of C
K Equivalent of H for $\omega \sim 1$ (other $\omega \sim 1$ details are not shown)
B Fold line of the $n = 2$ oscillation, meeting line H at point C
S Symmetry–breaking line of the oscillating $n = 2$ solution
F Crisis after short cascade via **oscillating chaos**: end of all oscillations
J Saddle–node creation of the $n = 1$ rotating orbit
A First period–doubling flip of the $n = 1$ rotating orbit
G Crisis after short cascade via **rotating chaos**: end of all rotations
I Restabilization of the $n = 1$ rotating orbit

Some over–laid zones

- (red) zone where the standing state is stable
- (blue) two zones where $n = 3$ orbits exist
- (grey) zone where the $n = 1$ rotation is again stable

Parametric Pendulum

$$\ddot{\theta} + \gamma\dot{\theta} + (1 + p\cos(\omega\tau))\sin\theta = 0$$

$$\ddot{\theta} + (1 + p\cos(\omega\tau))(\theta - \frac{1}{6}\theta^3) = 0$$

$$\ddot{\theta} + \bar{\omega}^2\theta - \frac{1}{6}\bar{\omega}^2\theta^3 = -\epsilon\cos\bar{\tau}\theta + \frac{1}{6}\epsilon\cos\bar{\tau}\theta^3$$

$$\bar{\tau} = \omega\tau,\ \bar{\omega} = 1/\omega \text{ and } \epsilon = p/\omega^2$$

$$\theta = \epsilon\theta_0 + \epsilon^2\theta_1 + \epsilon^3\theta_2 + \cdots.$$

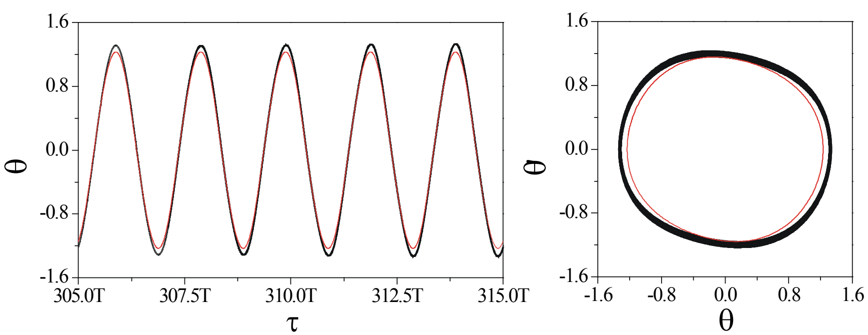
$$\epsilon^1: \quad \ddot{\theta}_0 + \bar{\omega}^2\theta_0 = 0,$$

$$\epsilon^2: \quad \ddot{\theta}_1 + \bar{\omega}^2\theta_1 = -\cos\bar{\tau}\theta_0,$$

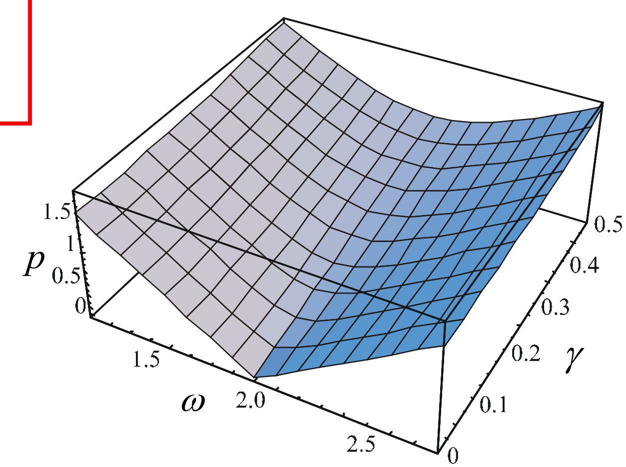
$$\epsilon^3: \quad \ddot{\theta}_2 + \bar{\omega}^2\theta_2 = -\cos\bar{\tau}\theta_1 + \frac{1}{6}\bar{\omega}^2\theta_0^3,$$

$$\vdots \qquad \vdots \qquad \vdots$$

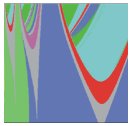
$$\theta = \sqrt{\epsilon}\alpha\cos(\frac{1}{2}\omega\tau + \frac{1}{2}\phi) + \frac{1}{16}\sqrt{\epsilon}\alpha p\cos(\frac{3}{2}\omega\tau + \frac{1}{2}\phi) - \frac{1}{192}\sqrt{\epsilon^3}\alpha^3\cos(3\tau + 3\phi),$$



$\gamma = 0.1,\ \omega = 1.9,\ p = 0.3$



Strutt-Ince diagram



Parametric Pendulum

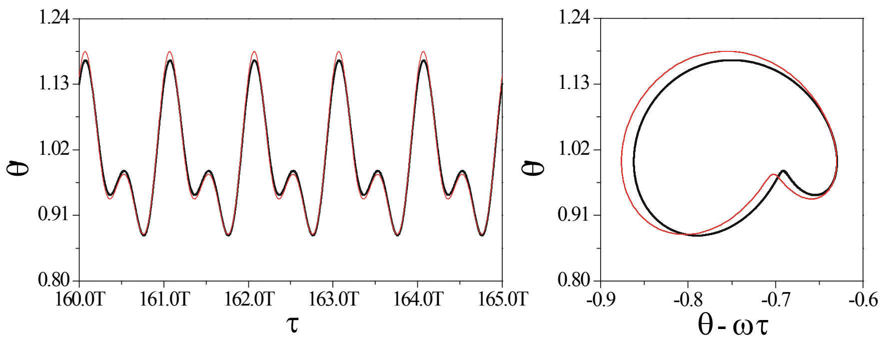
$$\ddot{\theta} + \epsilon\tilde{\gamma}\dot{\theta} + (\epsilon + \epsilon p \cos\tilde{\omega}\tilde{\tau})\sin\theta = 0$$

$$\tilde{\tau} = \frac{\tau}{\sqrt{\epsilon}}, \quad \tilde{\gamma} = \frac{\gamma}{\sqrt{\epsilon}}, \quad \boxed{\tilde{\omega} = \sqrt{\epsilon}\omega}$$

$$\theta = \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \cdots.$$

$$\theta = \tilde{\omega}\tilde{\tau} + \frac{\epsilon}{\tilde{\omega}^2}\sin(\tilde{\omega}\tilde{\tau} + \beta) + \frac{\epsilon p}{8\tilde{\omega}^2}\sin(2\tilde{\omega}\tilde{\tau} + \beta) + \beta,$$

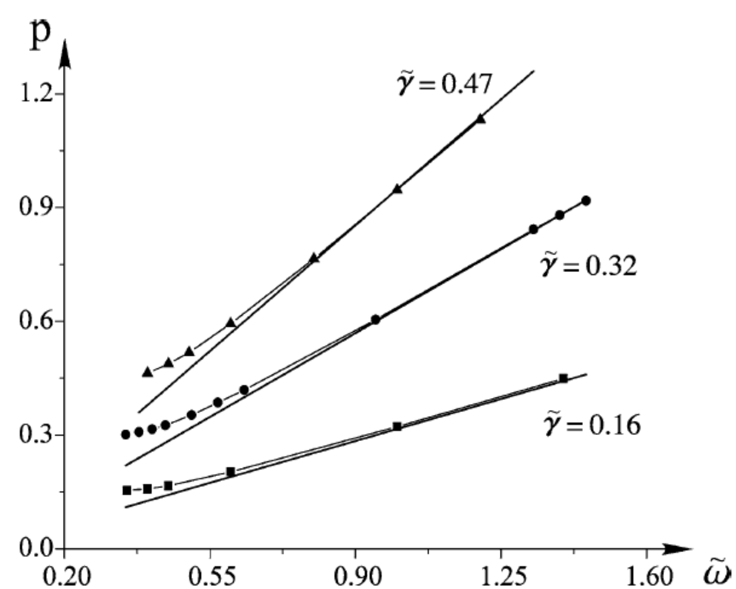
where $\beta = \arcsin(-2\tilde{\omega}\tilde{\gamma}/p)$



$\gamma = 0.32, \ \omega = 3.2, \ \ p = 3$

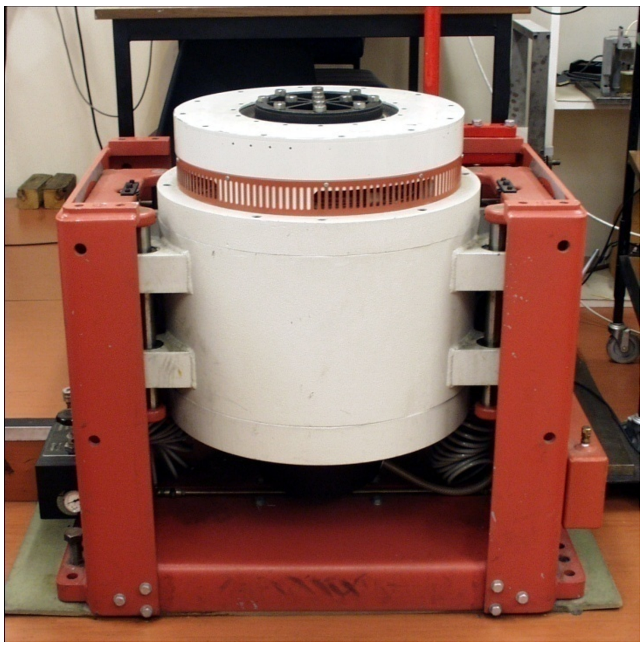
Comparison with numerical results

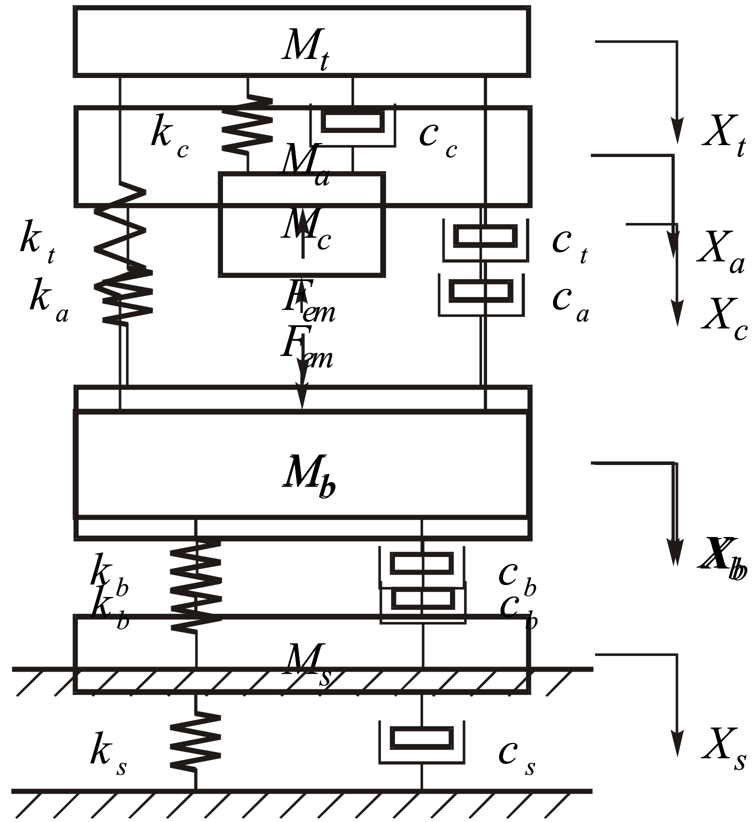
$$p \geq 2\tilde{\omega}\tilde{\gamma}$$



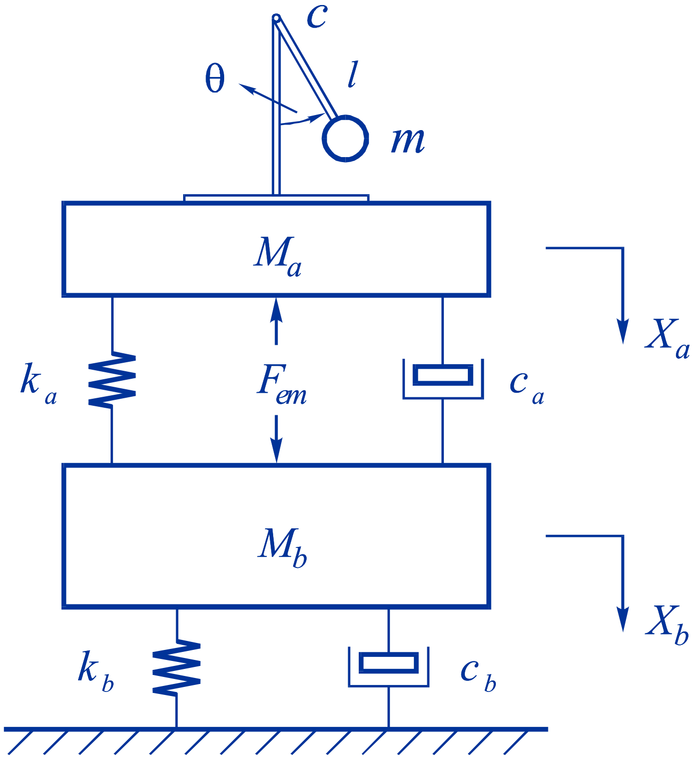
Nonlinear Dynamics (2007) **47**(1-3) 311-320

Shaker – Pendulum Interactions





Shaker – Pendulum Interactions



$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial \mathcal{D}}{\partial \dot{q}_i} = Q_i$$

$$\theta : \qquad ml\ddot{\theta} + cl\dot{\theta} + mg\sin\theta = m\ddot{X}_a \sin\theta;$$

$$X_a : \qquad (M_a + m)\ddot{X}_a + c_a(\dot{X}_a - \dot{X}_b) + k_a(X_a - X_b)$$
$$= (M_a + m)g + ml\ddot{\theta}\sin\theta + ml\dot{\theta}^2\cos\theta - F_{em};$$

$$X_b : \qquad M_b\ddot{X}_b + c_b\dot{X}_b - c_a(\dot{X}_a - \dot{X}_b) + k_bX_b$$
$$-k_a(X_a - X_b) = M_bg + F_{em};$$

$$I : \qquad RI + L\frac{dI}{dt} - \mathcal{K}(\dot{X}_a - \dot{X}_b) = E_0\cos(\Omega t),$$

where $F_{em} = \mathcal{K}I$

Shaker – Pendulum Interactions

$$\theta'' + \gamma\theta' + \sin\theta = x_a'' \sin\theta$$

$$\mu_2 x_b'' + \gamma(\chi_1 + \chi_2)x_b' + (\kappa_1 + \kappa_2)x_b = \gamma\chi_1 x_a' + \kappa_1 x_a + \mu_2 + \kappa_i i$$

$$(\mu_1 + \cos^2\theta)x_a'' + \gamma\chi_1(x_a' - x_b') + \kappa_1(x_a - x_b)$$
$$= -\kappa_i i + \mu_1 + \cos^2\theta - \gamma\theta' \sin\theta + \theta'^2 \cos\theta$$

$$\zeta i' + ri = \kappa_v(x_b' - x_a') + e_0 \cos\omega\tau.$$

Nondimensional variables:

$$\tau = \omega_n t, \qquad x_a = \frac{X_a}{l}, \qquad x_b = \frac{X_b}{l}, \qquad i = \frac{I}{I_0}$$

Nondimensional parameters:

$$\omega_n = \sqrt{\frac{g}{l}}, \quad \omega = \frac{\Omega}{\omega_n} \quad \gamma = \frac{c}{m\omega_n}, \quad \mu_1 = \frac{M_a}{m}, \quad \mu_2 = \frac{M_b}{m}, \quad \kappa_1 = \frac{k_a}{m\omega_n^2}, \quad \kappa_2 = \frac{k_b}{m\omega_n^2},$$

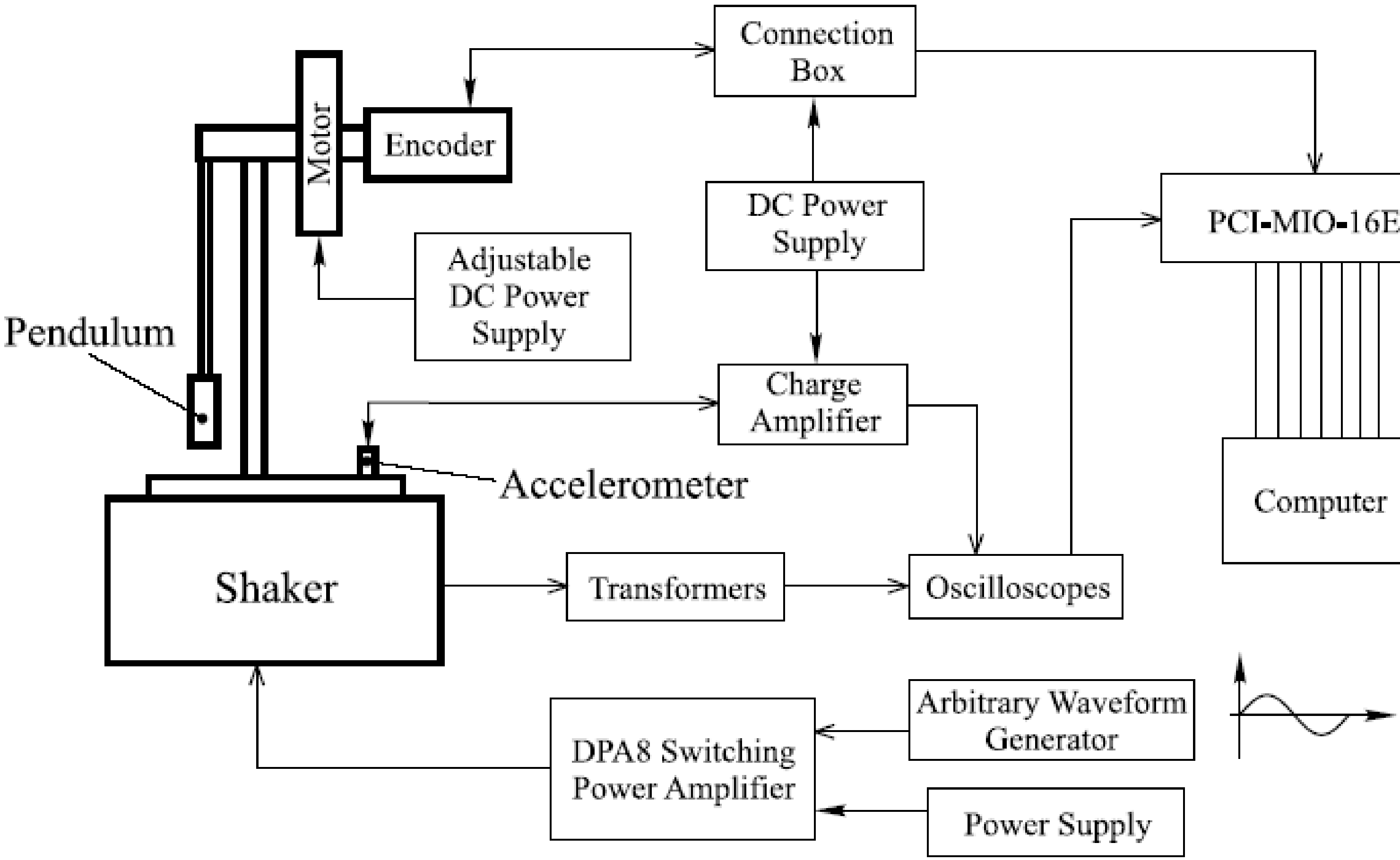
$$\chi_1 = \frac{c_a}{c}, \quad \chi_2 = \frac{c_b}{c}, \quad \kappa_i = \frac{\mathcal{K} I_0}{ml\omega_n^2}, \quad \kappa_v = \frac{\mathcal{K} l\omega_n}{R_0 I_0}, \quad \zeta = \frac{L\omega_n}{R_0}, \quad r = \frac{R}{R_0}, \quad e_0 = \frac{E_0}{R_0 I_0}.$$

Shaker – Pendulum Interactions

Mass of the pendulum: 0.854kg
Mass of the pendulum support: 51.6kg
Length of the pendulum rod: 0.3166m
Mass of the armature assembly: 15.88kg
Mass of the shaker body: 820kg
Resistance of the armature coil: 0.3 Ω
Impedance of the armature coil under a 1kHz AC: 15 ~ 18 Ω

Pendulum	$m = 0.854$ kg	$l = 0.3166$ m	$c = ?$
Shaker	$M_a = 67.5$ kg	$k_a = ?$	$c_a = ?$
	$M_b = 820$ kg	$k_b = ?$	$c_b = ?$
	$R = 0.3\ \Omega$	$L = 2.626\ 10^{-3}$ H	$\mathcal{K} = ?$

Experimental Rig

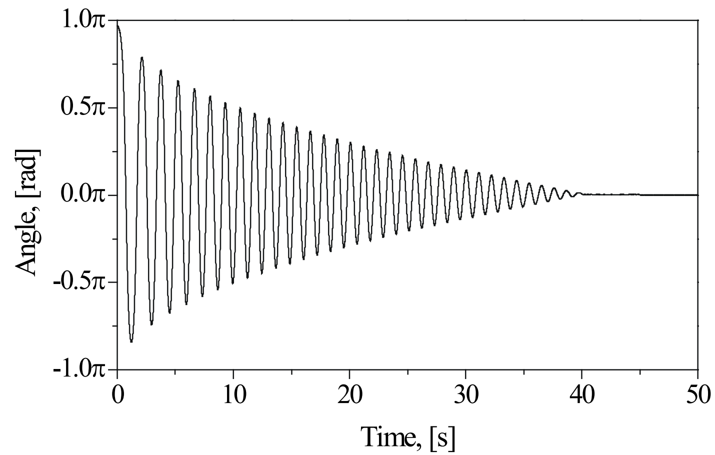


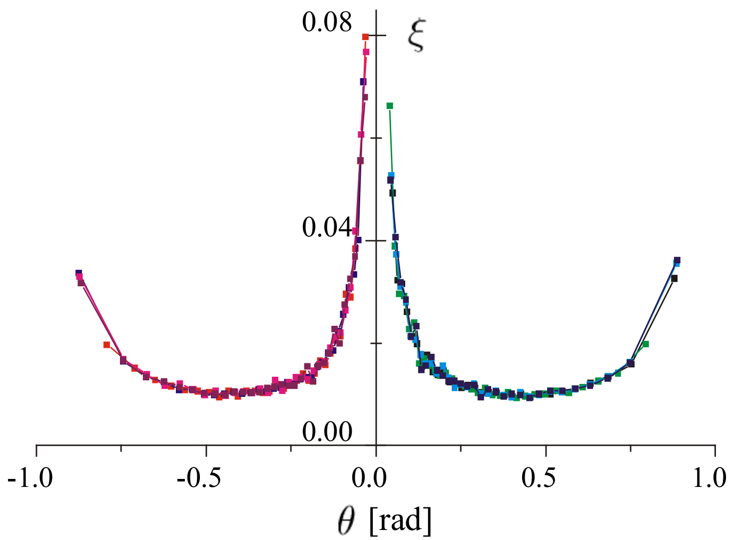
Shaker – Pendulum Interactions

Logarithmic Decrement Method:

$$\delta = \frac{2\pi\varsigma}{\sqrt{1-\varsigma^2}} = \frac{1}{n}\ln\left(\frac{x_0}{x_n}\right)$$

$$\varsigma = \frac{\delta}{\sqrt{4\pi^2+\delta^2}}$$





ξ = 0.01

$$c = 2m\xi\sqrt{\frac{g}{l}} = 0.095$$ kg/s

Shaker – Pendulum Interactions

$$\omega_{na} = \frac{2\pi f_a}{\sqrt{1-\xi_a^2}} = 35.50\mathrm{rad/s},$$

$$\omega_{nb} = \frac{2\pi f_b}{\sqrt{1-\xi_b^2}} = 17.26\mathrm{rad/s}.$$

$$c_a = 2M_a\omega_{na}\xi_a = 534\mathrm{kg/s}, \qquad k_a = M_a\omega_{na}^2 = 86176\mathrm{N/m},$$

$$c_b = 2M_b\omega_{nb}\xi_b = 679\mathrm{kg/s}. \qquad k_b = M_b\omega_{nb}^2 = 244284\mathrm{N/m}.$$

m	0.854 kg	l	0.3166 m	c	0.0475 kg/s
M_a	68.38 kg	K_a	86175.9 kg/s^2	C_a	534.05 kg/s
M_b	820 kg	K_b	244284 kg/s^2	C_b	679.35 kg/s
R	0.3 Ω	L	2.626×10^{-3} H	$\mathcal{K}$	130 N/A

ZAMM (2007) **87**(2), 172-187

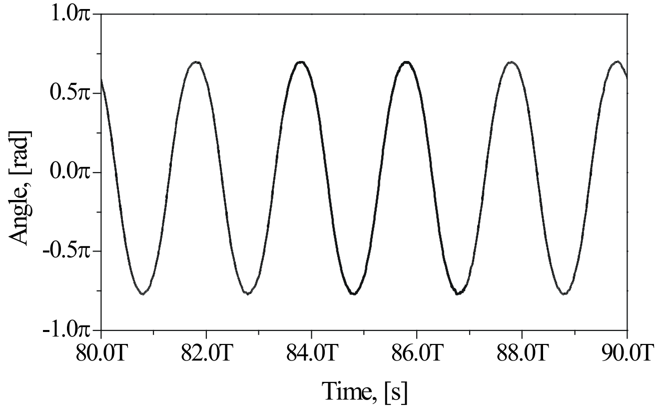
Shaker – Pendulum Interactions

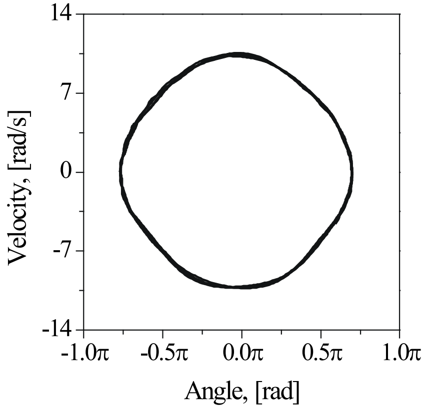
Oscillations, Rotations and Transient Tumbling Chaos

Oscillations

(a) $E_0 = 14.25\text{V}$

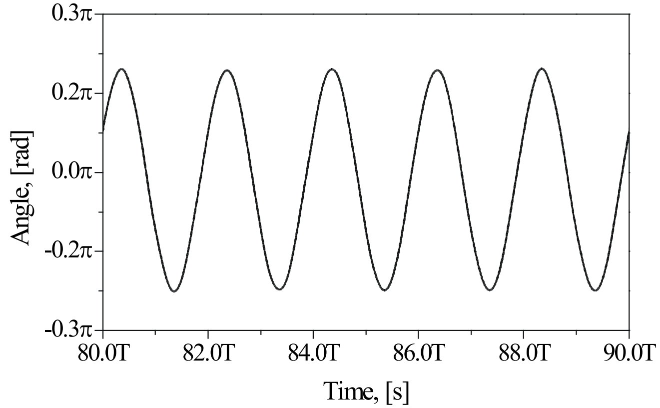
$f = 1.24\text{Hz}$

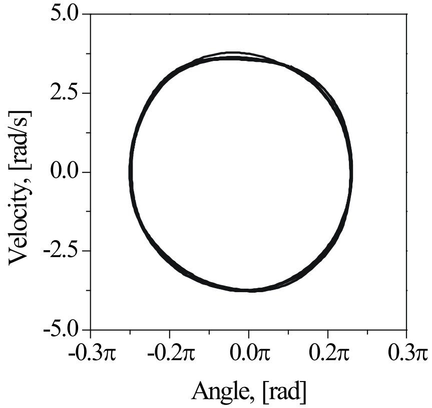




(b) $E_0 = 18.82\text{V}$

$f = 1.86\text{Hz}$



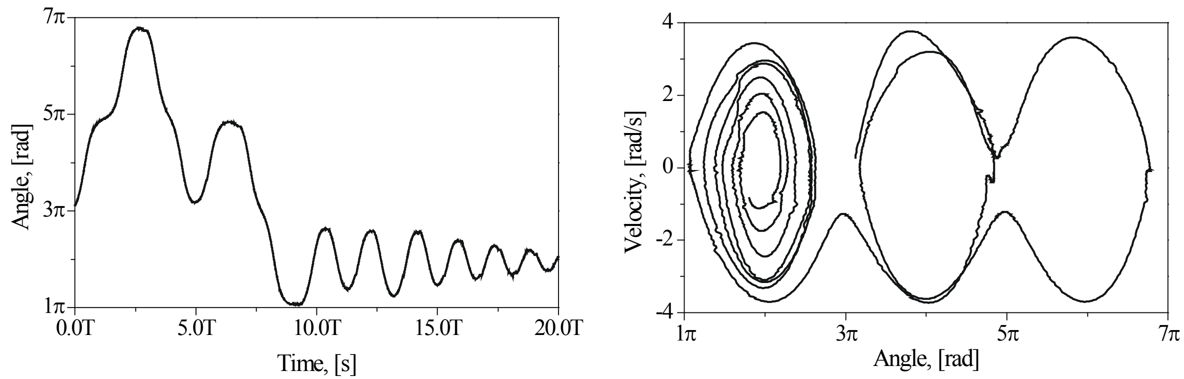


Shaker – Pendulum Interactions

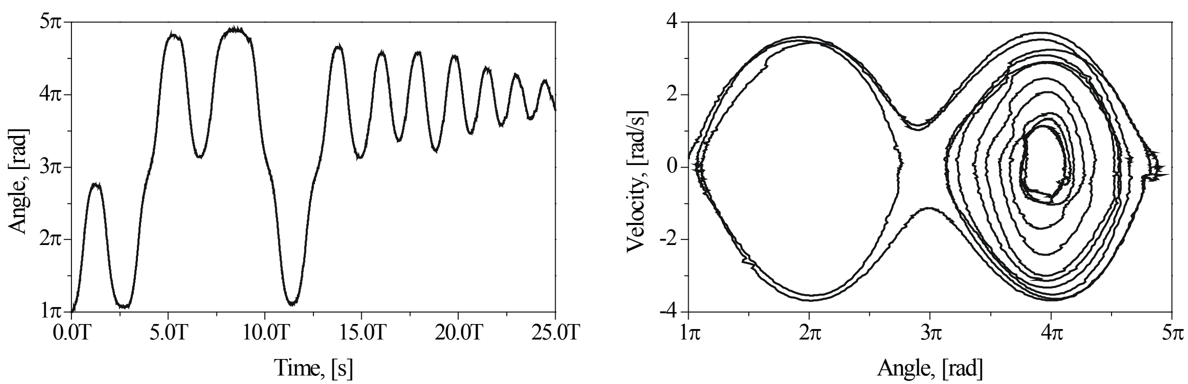
Experimental Studies
— Oscillations, Rotations and Transient Tumbling Chaos

Transient Tumbling Chaos $(f = 1.24\text{Hz})$

(a)



(b)



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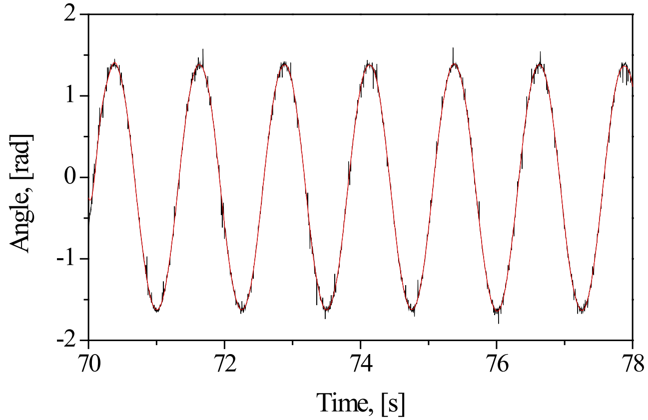
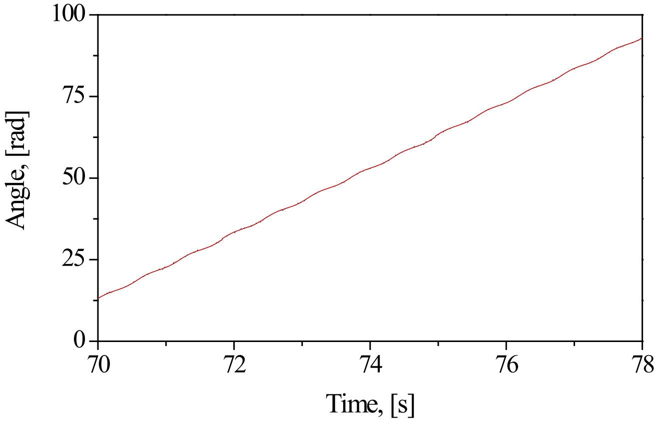
Shaker – Pendulum Interactions

Experimental Studies — Comparisons with Numerical Results

Oscillations

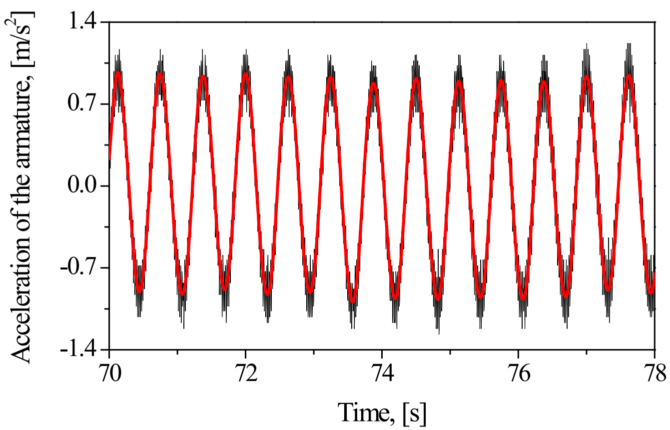


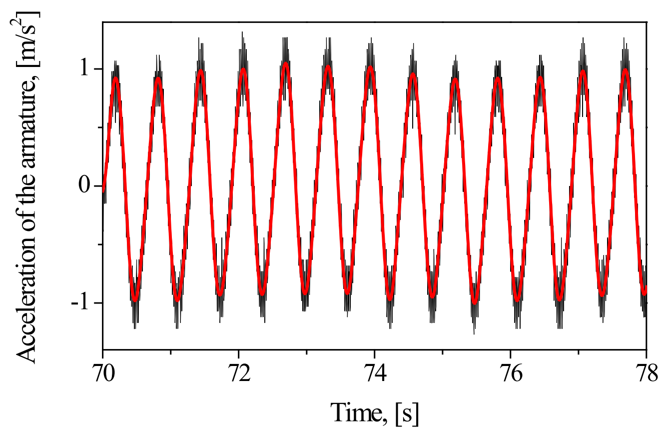
Rotations



Experiments

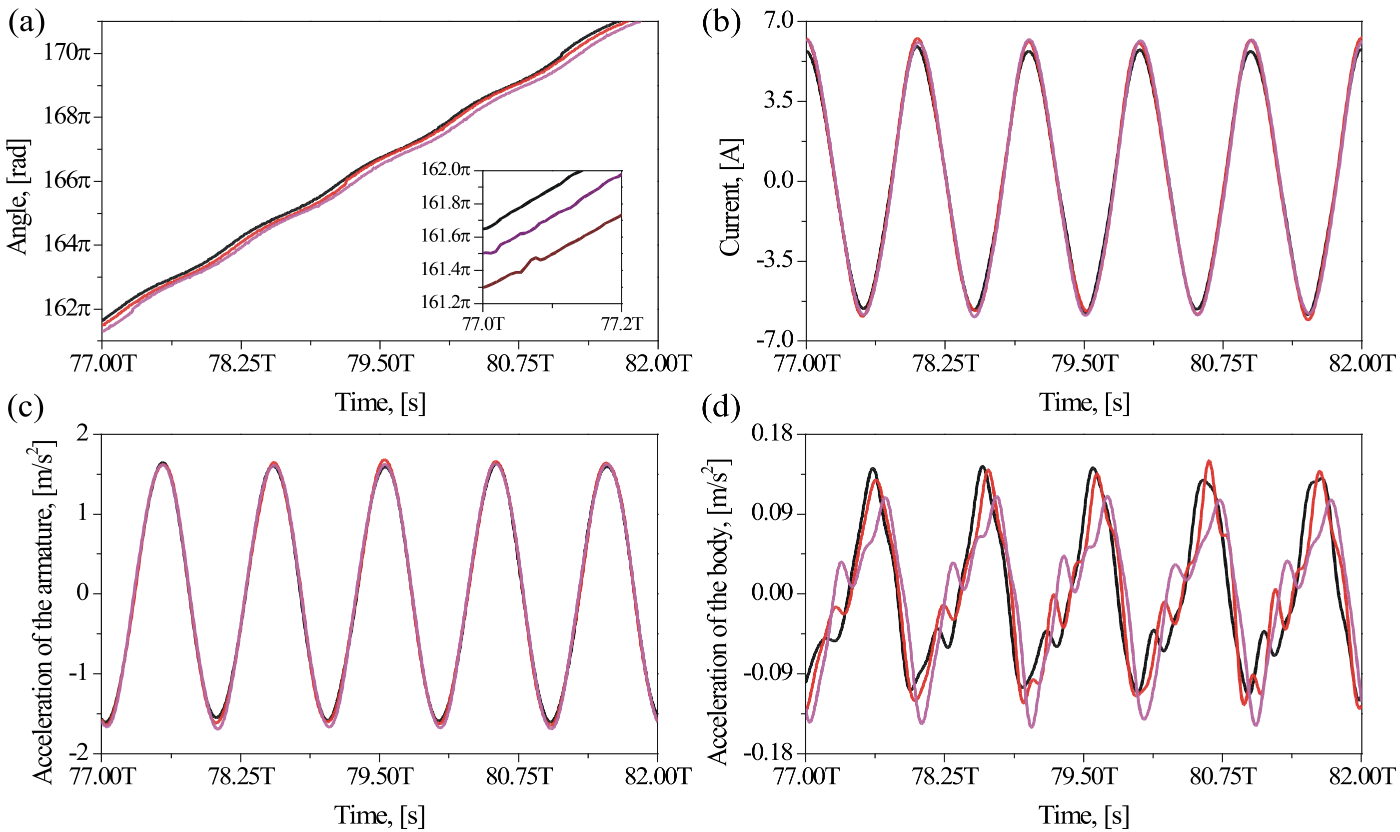
Theory





Shaker – Pendulum Interactions

Rotations With and Without Power Take Off

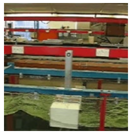


Rotation without energy output

Rotation with energy output of 16 W

Rotation with energy output of 70 W

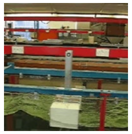
Experiments in Water Tank



Tumbling Chaos

1 4 9 5

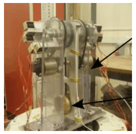
Experiments in Water Tank



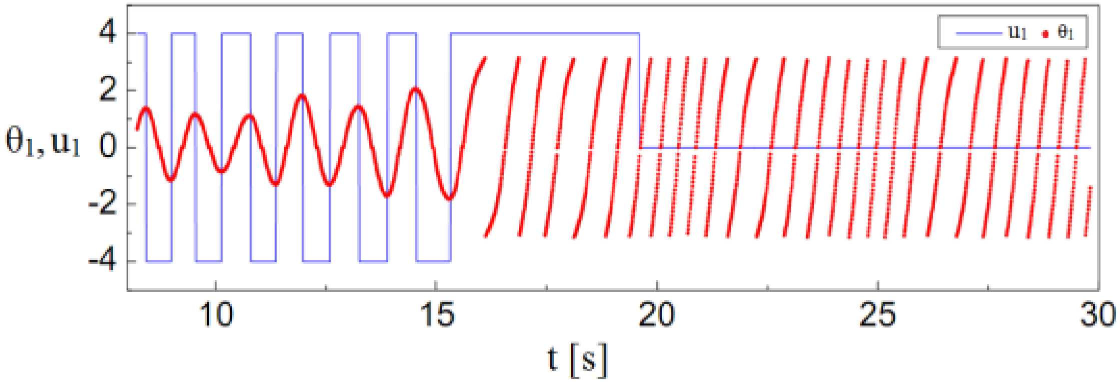
Stable Rotations

Experiments in Water Tank

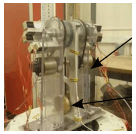
Stable Rotations for varying frequency and amplitude

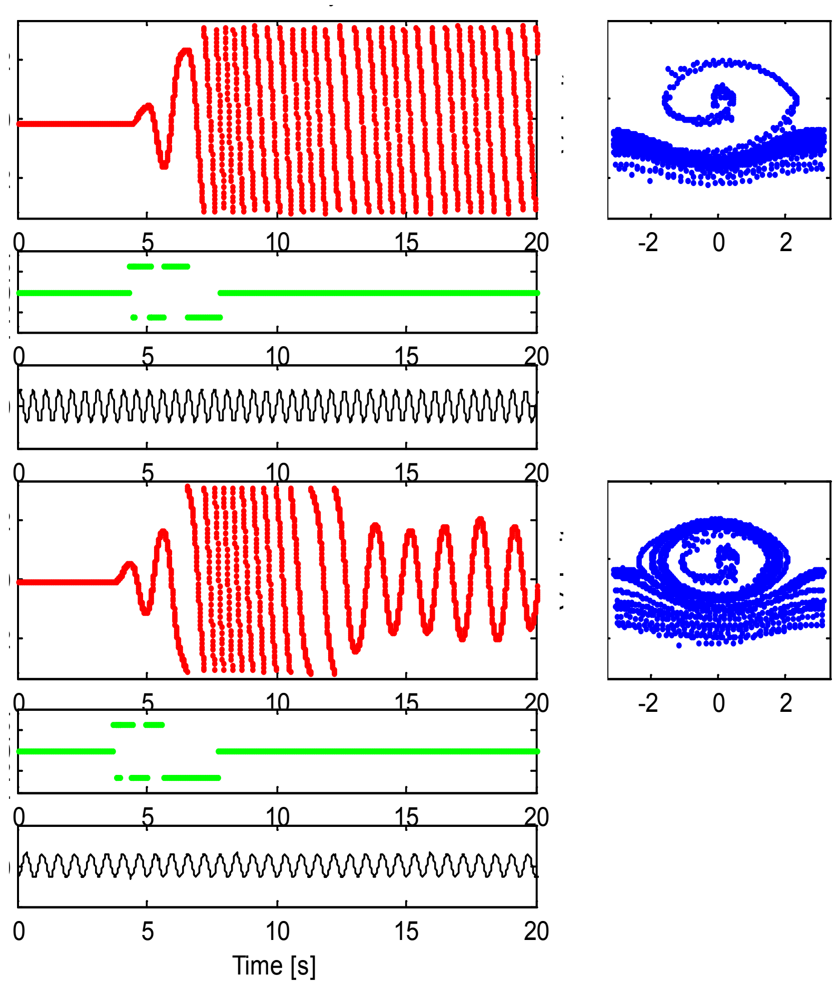


Control – Initialisation of Rotations



Initialisation of Rotations – Bang-Bang





$$\dot{\mathbf{x}}(t) = \mathbf{Q}(\mathbf{x}, t) + \mathbf{B}(t),$$

$$\mathbf{B}(t) = \mathbf{K}[(1 - R)\mathbf{S}_\tau - \mathbf{x}],$$

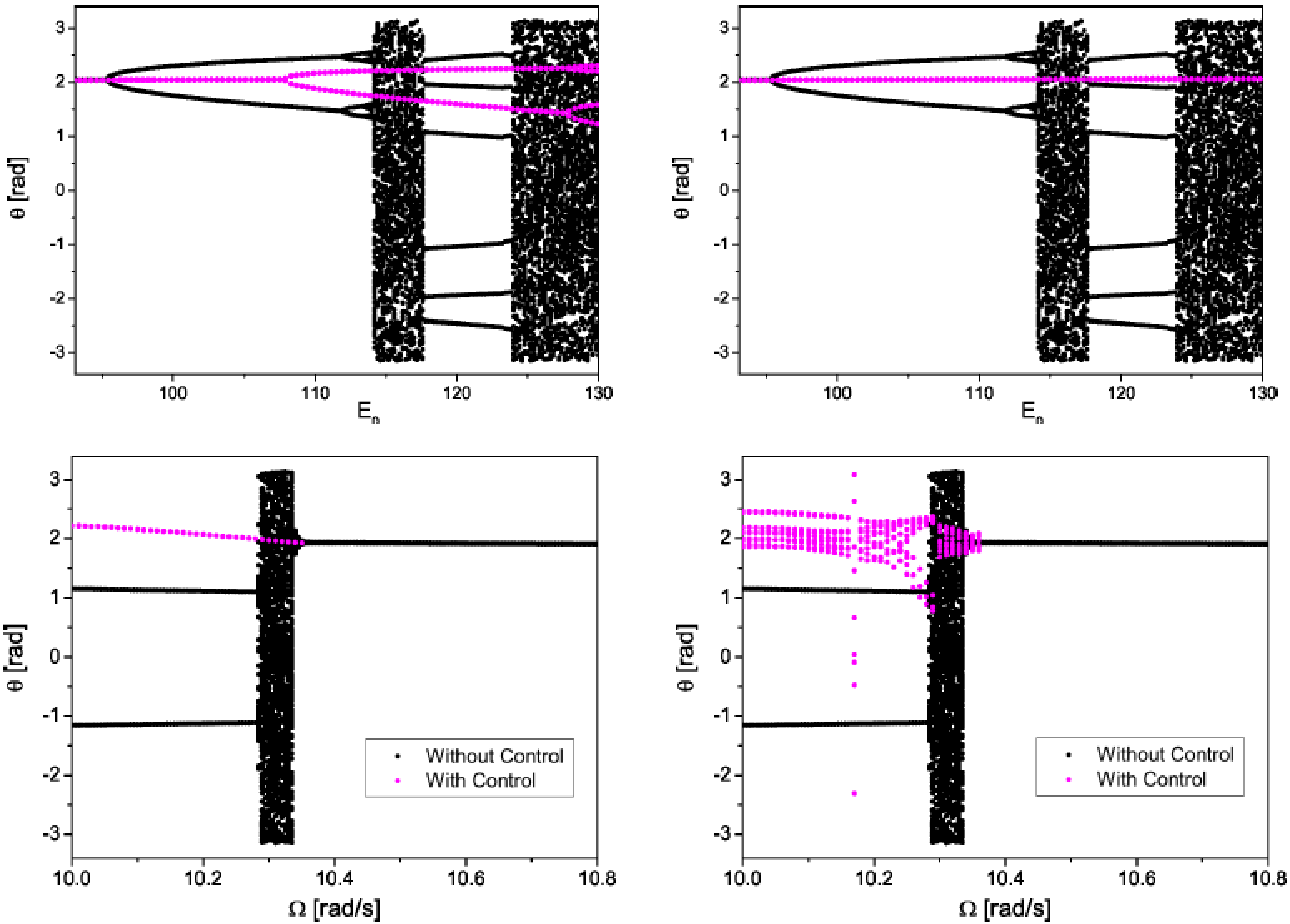
$$\mathbf{S}_\tau = \sum_{m=1}^{N_\tau} R^{m-1}\mathbf{x}_{m\tau},$$

$$\mathbf{x}_{m\tau} = \mathbf{x} - m\tau\dot{\mathbf{x}}.$$

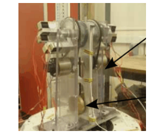
$$\dot{\mathbf{x}}(t) = \mathbf{Q}(\mathbf{x}, t) + \mathbf{B}(t, \mathbf{x}, \mathbf{x}_\tau, \mathbf{x}_{2\tau}, \mathbf{x}_{3\tau}).$$

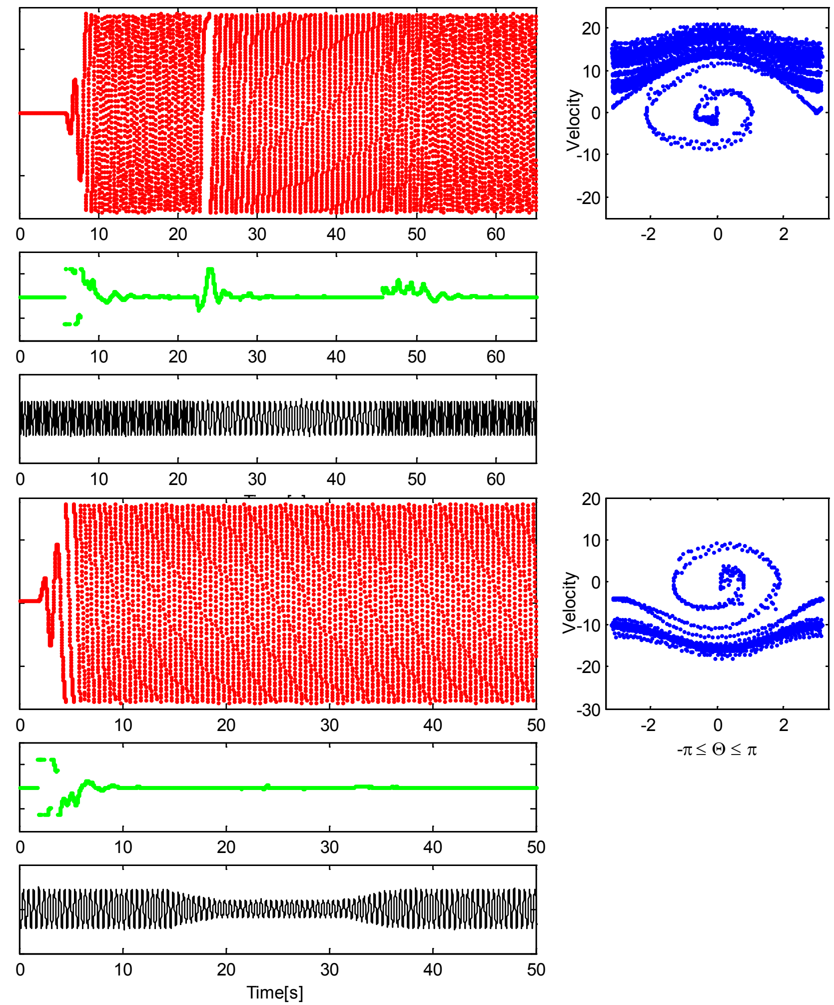
IJBC (2012) **22**(5), 1250111

Bifurcation Control

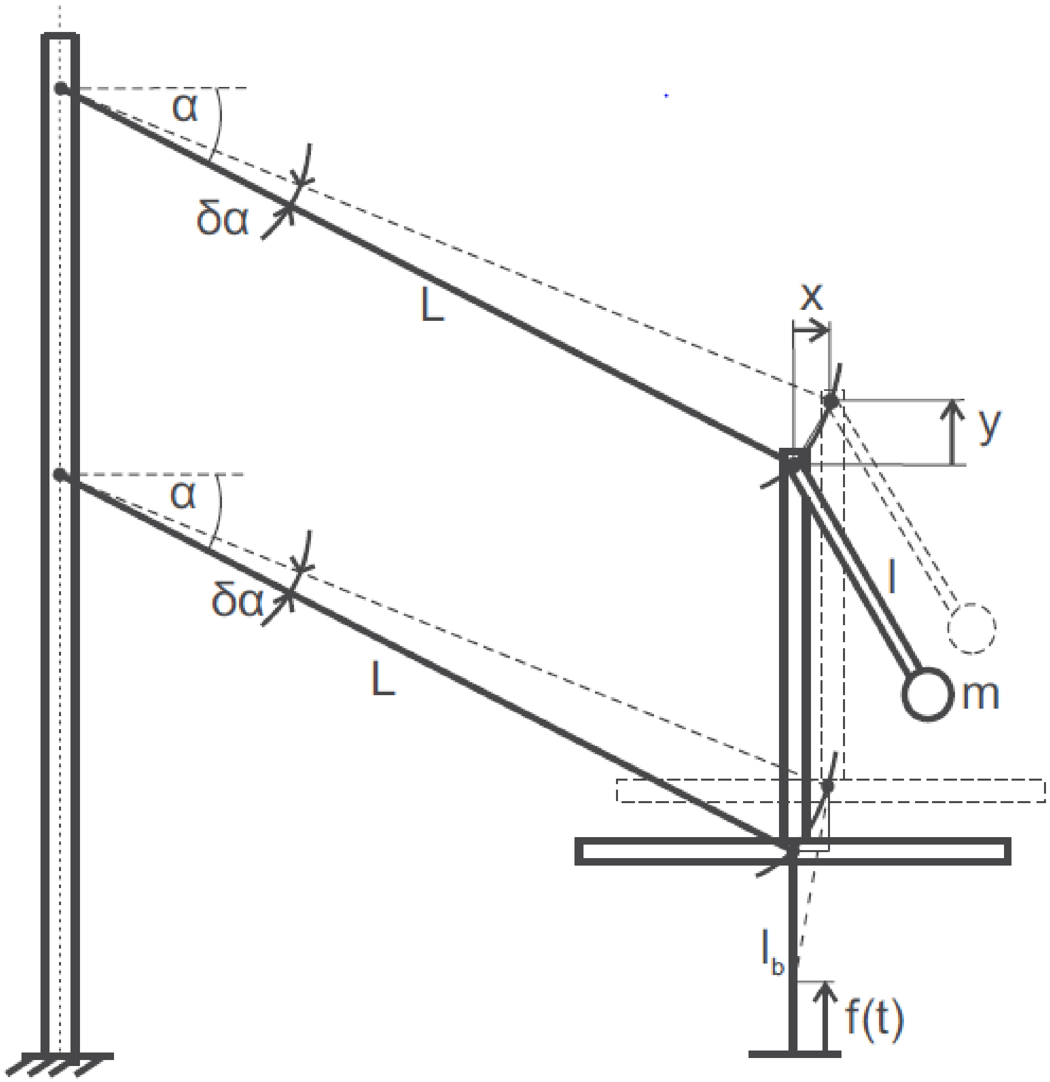


Maintaining Rotations – Delayed Feedback

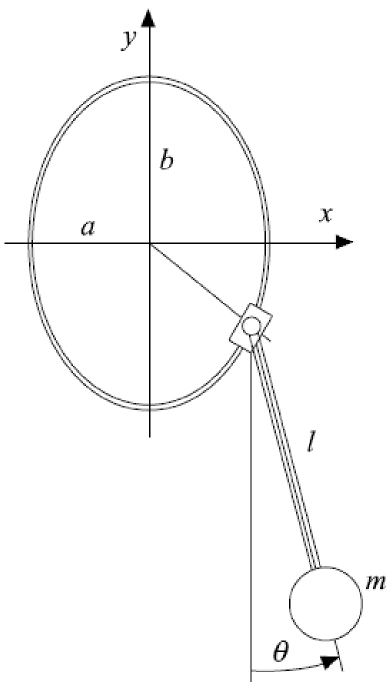




Planar Excitation



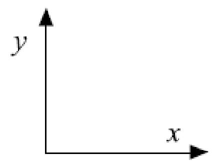
Planar Excitation



Eliptical Excitation

$$\ddot{\theta} + \gamma\dot{\theta} + (1 + p\cos(\omega t))\sin\theta + ep\sin(\omega t)\cos\theta = 0$$

Planar Excitation



$$\ddot{\theta} + \gamma\dot{\theta} + \omega f_y(t)\sin\theta + \omega f_x(t)\cos\theta = 0$$

$$\phi(t) = \theta(t) - \omega t$$

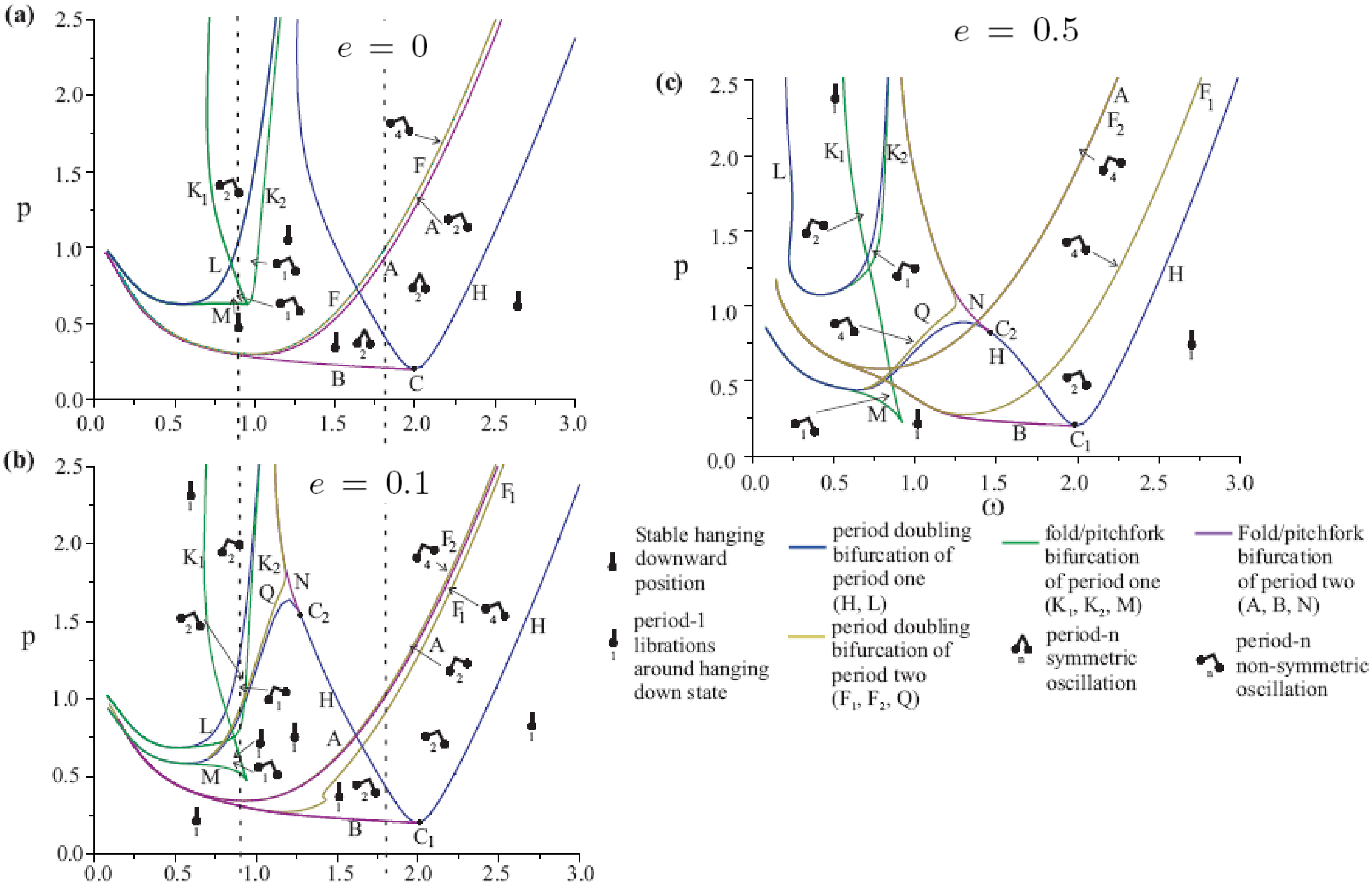
$$\ddot{\phi} + \frac{\gamma}{\sqrt{\omega}}\dot{\phi} + \gamma + \frac{1}{2}\left[f_y^c - f_x^s\right]\sin\phi + \frac{1}{2}\left[f_y^s + f_x^c\right]\cos\phi = 0$$

$$f_{x,y}^c = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f_{x,y}(s)\cos(\omega s)\,\mathrm{d}s,$$

$$f_{x,y}^s = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f_{x,y}(s)\sin(\omega s)\,\mathrm{d}s$$

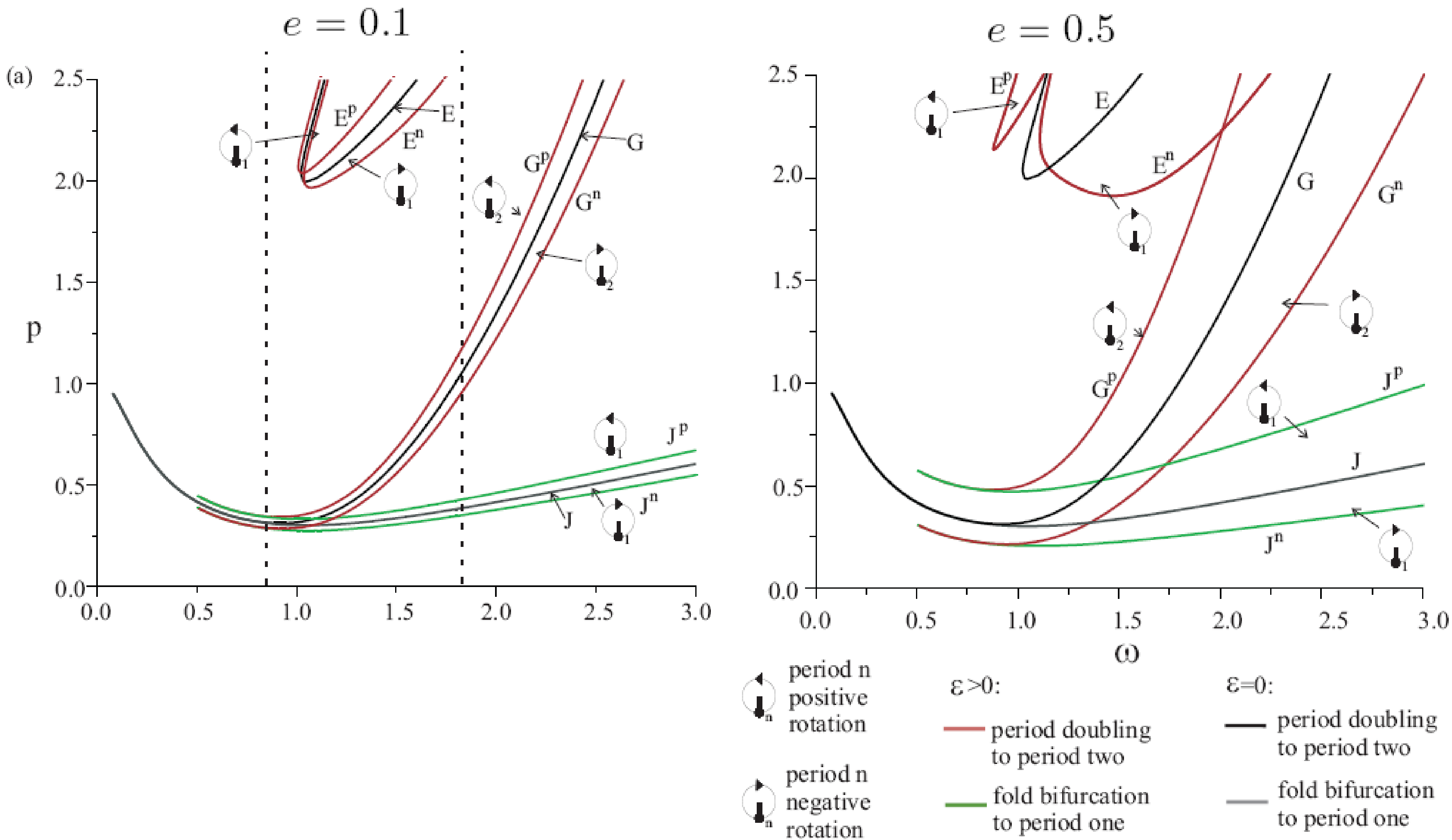
Planar Excitation

Oscillations



Planar Excitation

Rotations



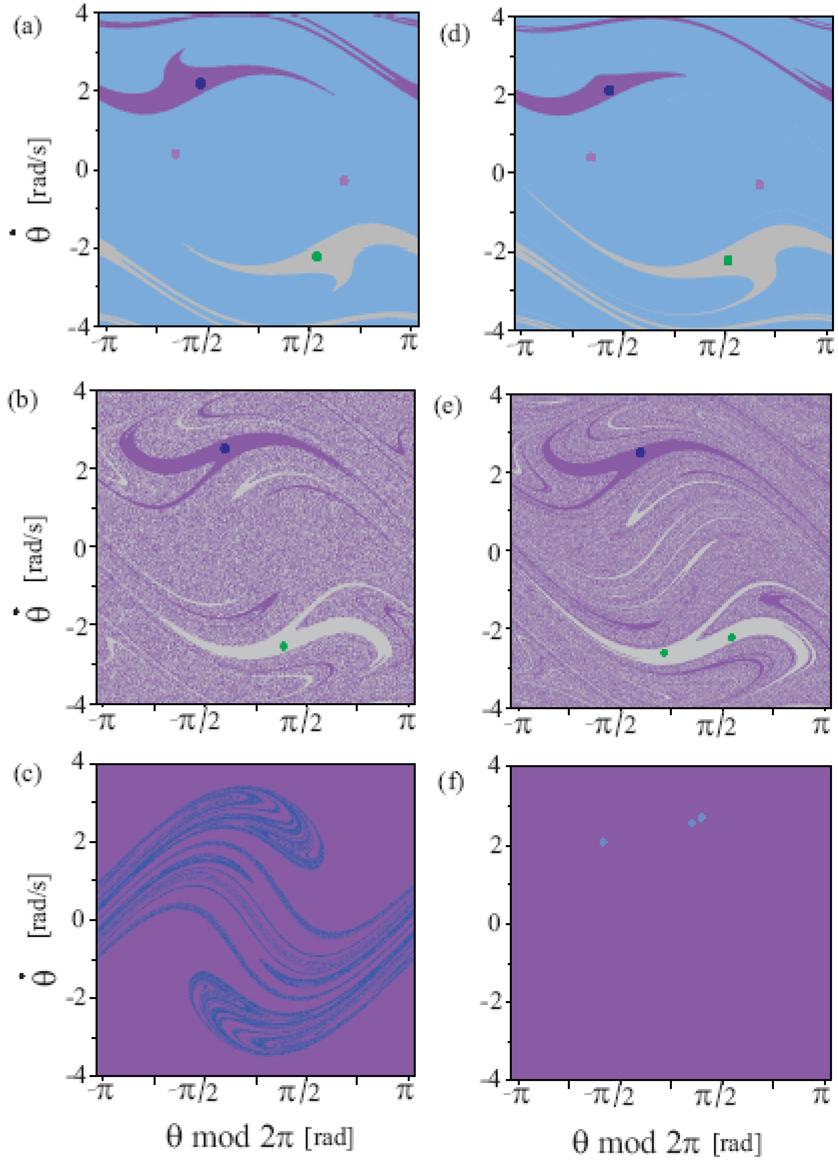
IJNM (2011) **46**, 436-442

Planar Excitation

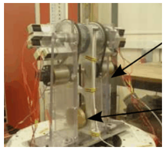
$e = 0,\ p = 0.5$ $e = 0.1,\ p = 0.5$

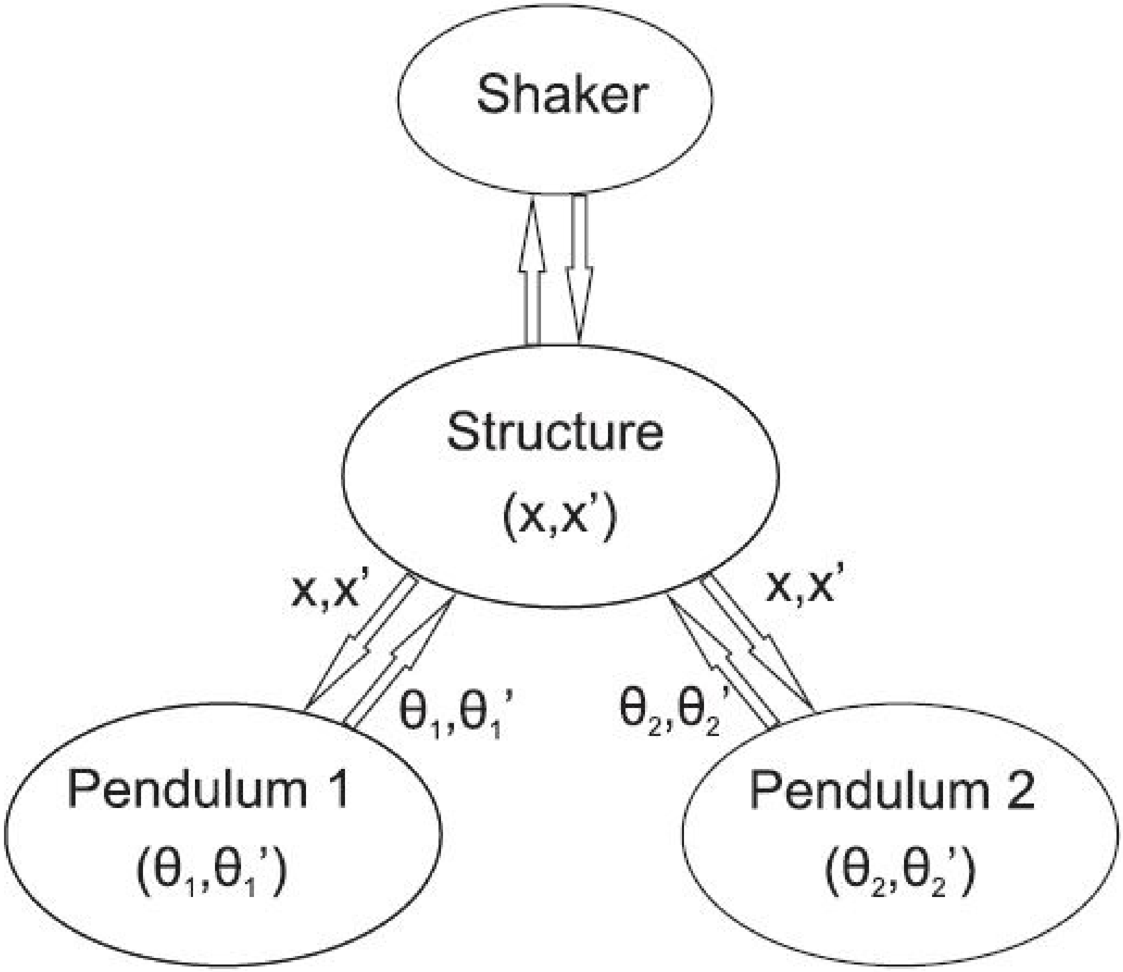
$e = 0,\ p = 1$ $e = 0.1,\ p = 1$

$e = 0,\ p = 1.4$ $e = 0,\ p = 1.4$

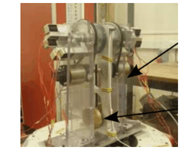


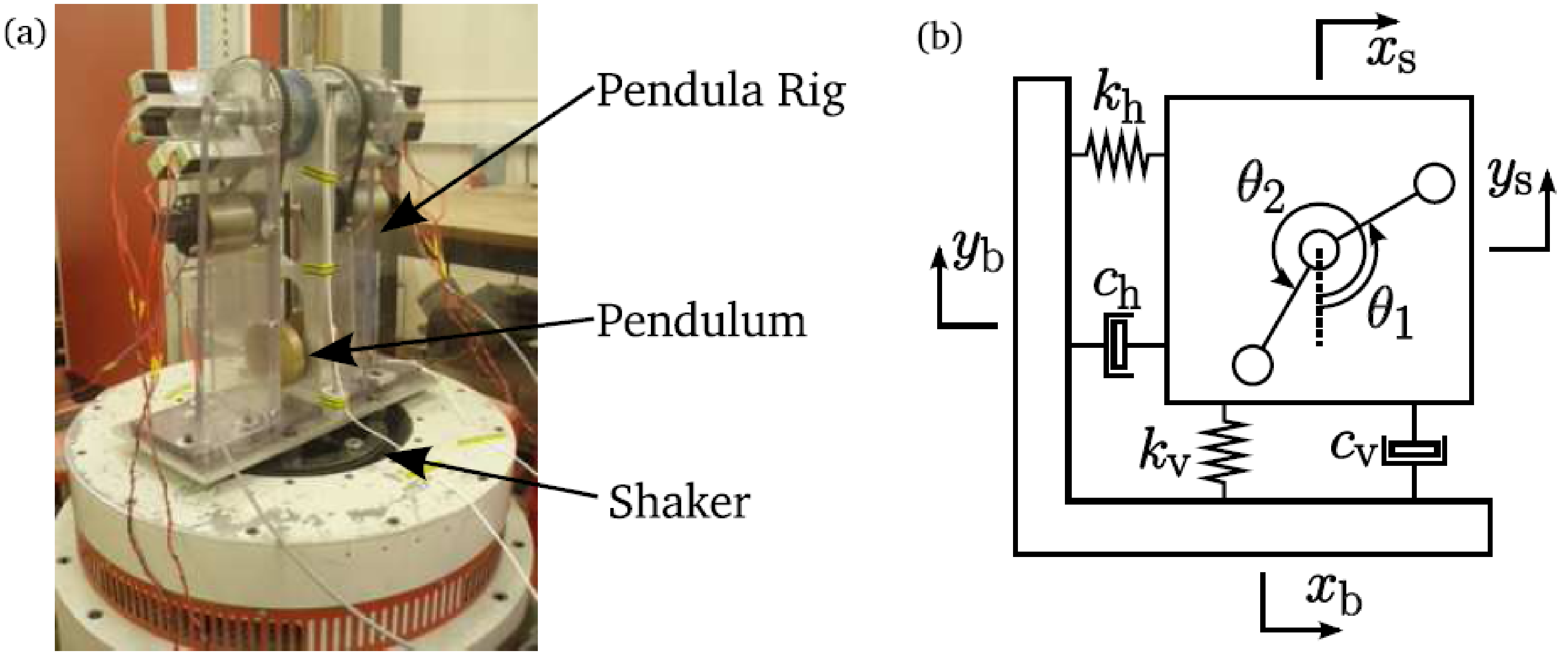
Pendula Systems





IJNM (2015) **71**, 84-94



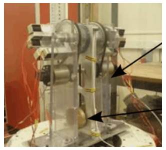


$$\theta_i : \; m_i l_i \ddot{\theta}_i + c_i l_i \dot{\theta}_i + m_i g \sin\theta_i = -m_i \ddot{x}_{\mathrm{s}} \cos\theta_i - m_i \ddot{y}_{\mathrm{s}} \sin\theta_i, \quad (i = 1, 2)$$

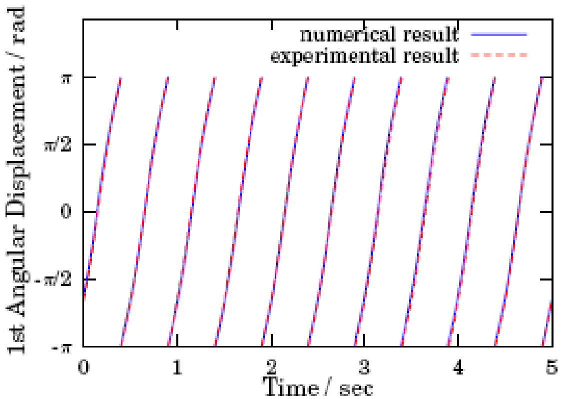
$$\begin{aligned} x_{\mathrm{s}} : \; & (m_{\mathrm{s}} + m_1 + m_2)\ddot{x}_{\mathrm{s}} + c_{\mathrm{h}}(\dot{x}_{\mathrm{s}} - \dot{x}_{\mathrm{b}}) + k_{\mathrm{h}}(x_{\mathrm{s}} - x_{\mathrm{b}}) \\ & = -m_1 l_1 (\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 \sin\theta_1) - m_2 l_2 (\ddot{\theta}_2 \cos\theta_2 - \dot{\theta}_2^2 \sin\theta_2), \end{aligned}$$

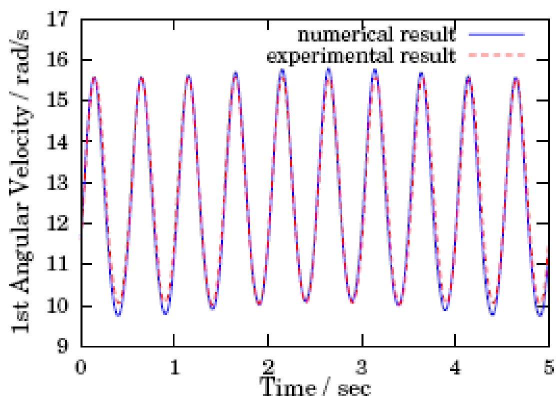
$$\begin{aligned} y_{\mathrm{s}} : \; & (m_{\mathrm{s}} + m_1 + m_2)\ddot{y}_{\mathrm{s}} + c_{\mathrm{v}}(\dot{y}_{\mathrm{s}} - \dot{y}_{\mathrm{b}}) + k_{\mathrm{v}}(y_{\mathrm{s}} - y_{\mathrm{b}}) \\ & = -m_1 l_1 (\ddot{\theta}_1 \sin\theta_1 + \dot{\theta}_1^2 \cos\theta_1) - m_2 l_2 (\ddot{\theta}_2 \sin\theta_2 + \dot{\theta}_2^2 \cos\theta_2), \end{aligned}$$

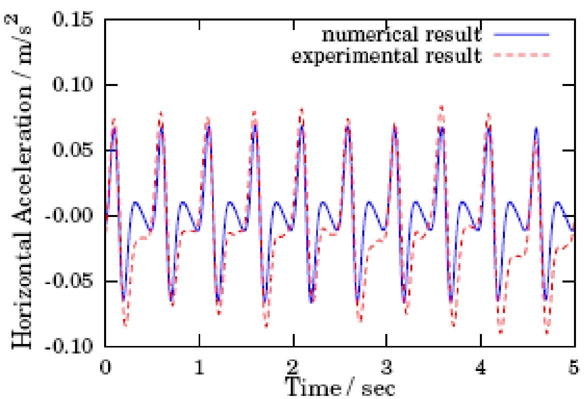
Pendula Systems

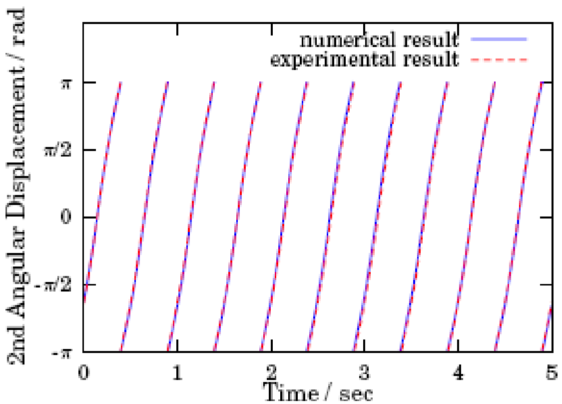


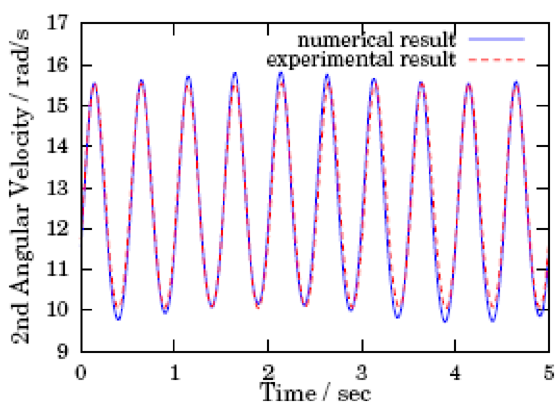
m_i	0.709 kg	c_1^{cc}	0.0509 kg/s	c_1^{cl}	0.0540 kg/s
l_i	0.271 m	c_2^{cc}	0.0423 kg/s	c_2^{cl}	0.0765 kg/s
m_{s}	11.7 kg	k_{h}	5.61×10^5 kg/s^2	c_{h}	6.22×10^2 kg/s
		k_{v}	2.20×10^8 kg/s^2	c_{v}	1.85×10^3 kg/s

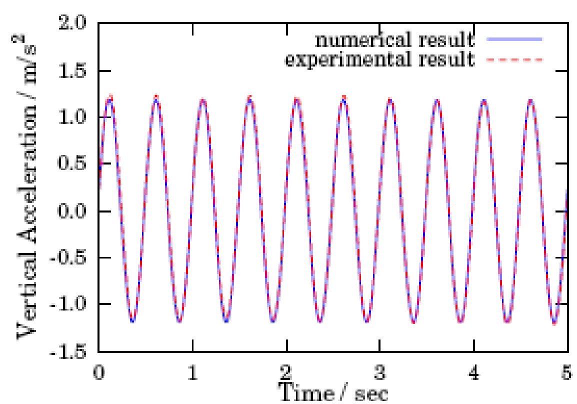




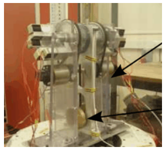


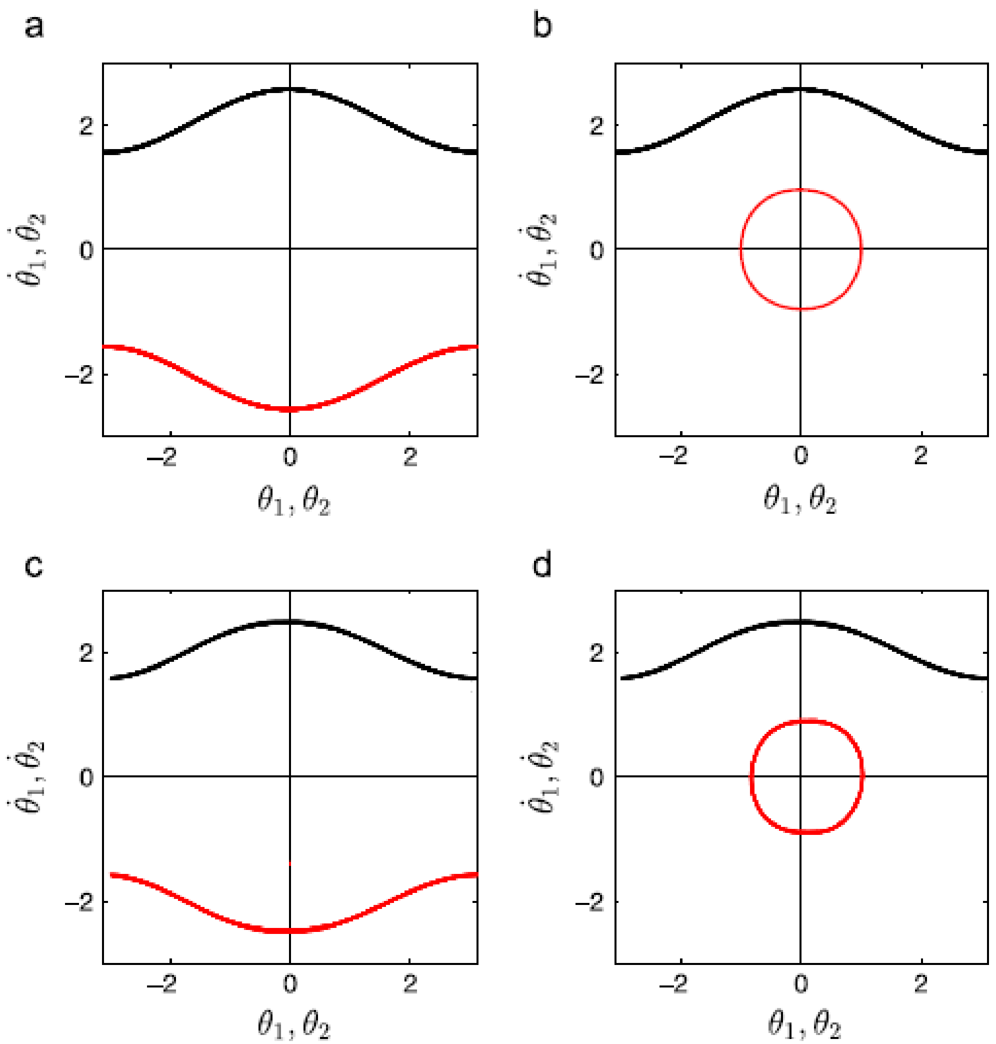




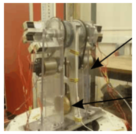


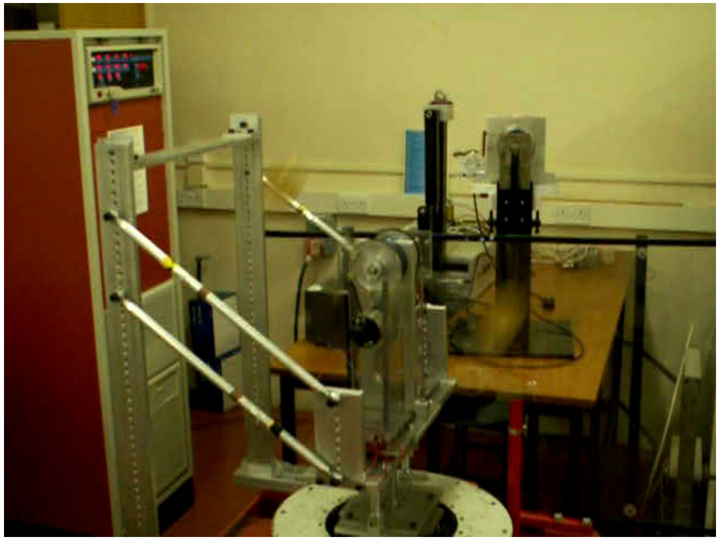
Pendula Systems

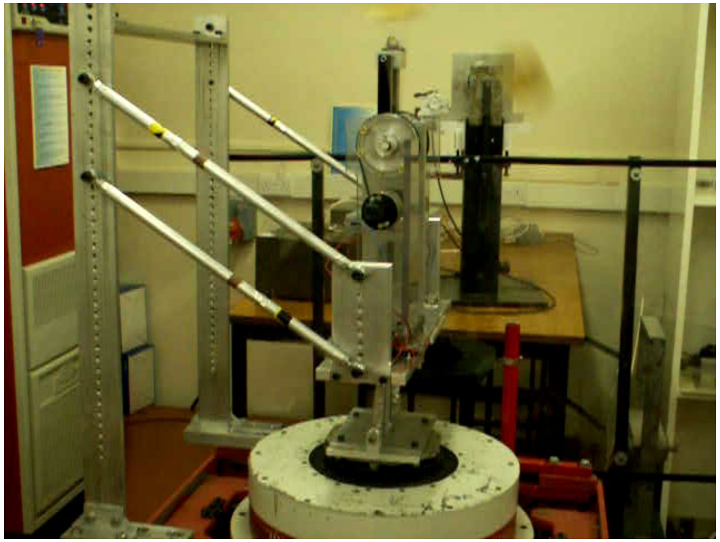




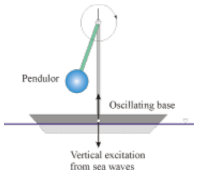
Pendula System - Synchronization







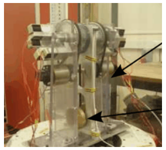
Technical Viability



Viability of the concept of wave energy extraction via a Parametric Pendulum

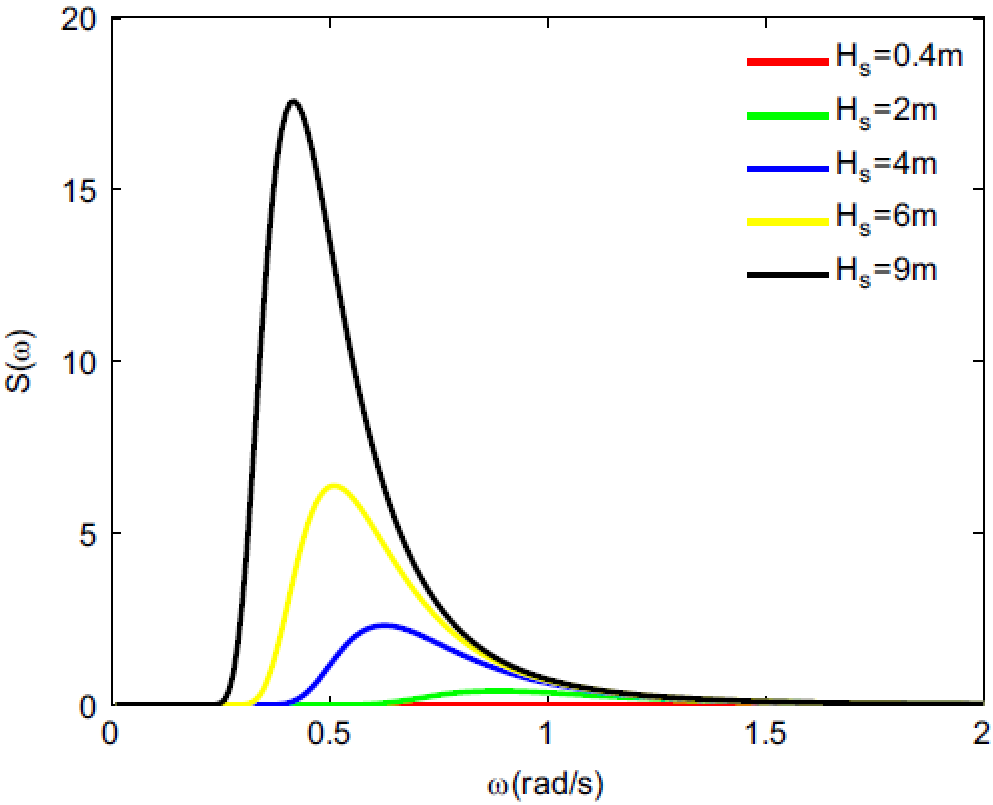
- small scale rig (1kg, 2.5 Hz) generates 160 W
- Power/mass ratio – 16 (0.35)
- scaled up extractor (10000 kg, 0.1 Hz) should give > 1.0 MW
- wind turbine technology can be fully utilized

Stochasticity of Excitation

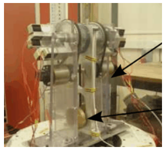


$$S(\omega) = \frac{Ag^2}{\omega^2} \exp\left[-B\left(\frac{(g/U)}{\omega}\right)^4 \right]$$

$$S(\omega) = \frac{8.1}{10^3}\frac{g^2}{\omega^5} \exp\left[-0.032\frac{(g/H_s)^2}{\omega^4} \right]$$



Stochasticity of Excitation



$$f(t)=\sqrt{2}\sum_{k=1}^{N}\sqrt{S_0(\omega_k)\Delta\omega}\cos(\omega_k t+\phi_k)$$

$$f(t)=\sqrt{2}\sum_{k=1}^{N}\sqrt{\frac{8.1}{10^3}\frac{g^2}{\omega_k^5}\exp\left[-0.032\frac{(g/H_s)^2}{\omega_k^4}\right]\Delta\omega}\cos(\omega_k t+\phi_k)$$

$$f(t)=\sqrt{2}\sum_{k=1}^{N}\sqrt{S_0(\omega_k)\Delta\omega_k}\cos(\omega_k t+\phi_k)$$

$$\Delta\omega_k=\omega_k-\omega_{k-1},\quad \Delta\omega_k\neq\Delta\omega_{k-1}$$

$$\int_{\omega_{k-1}}^{\omega_k}S(\omega)\,\mathrm{d}\omega=\int_{\omega_k}^{\omega_{k+1}}S(\omega)\,\mathrm{d}\omega$$

$$\int_{\omega_0}^{\omega_k}S(\omega)\,\mathrm{d}\omega=\frac{k}{N}\int_{\omega_0}^{\omega_N}S(\omega)\,\mathrm{d}\omega.$$

$$\omega_k=\left(\frac{0.032\left(\frac{g}{H_s}\right)^2}{\ln\frac{N}{k}+0.032\left(\frac{g}{H_s}\right)^2/\omega_n^4}\right)^{0.25}.$$

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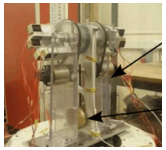
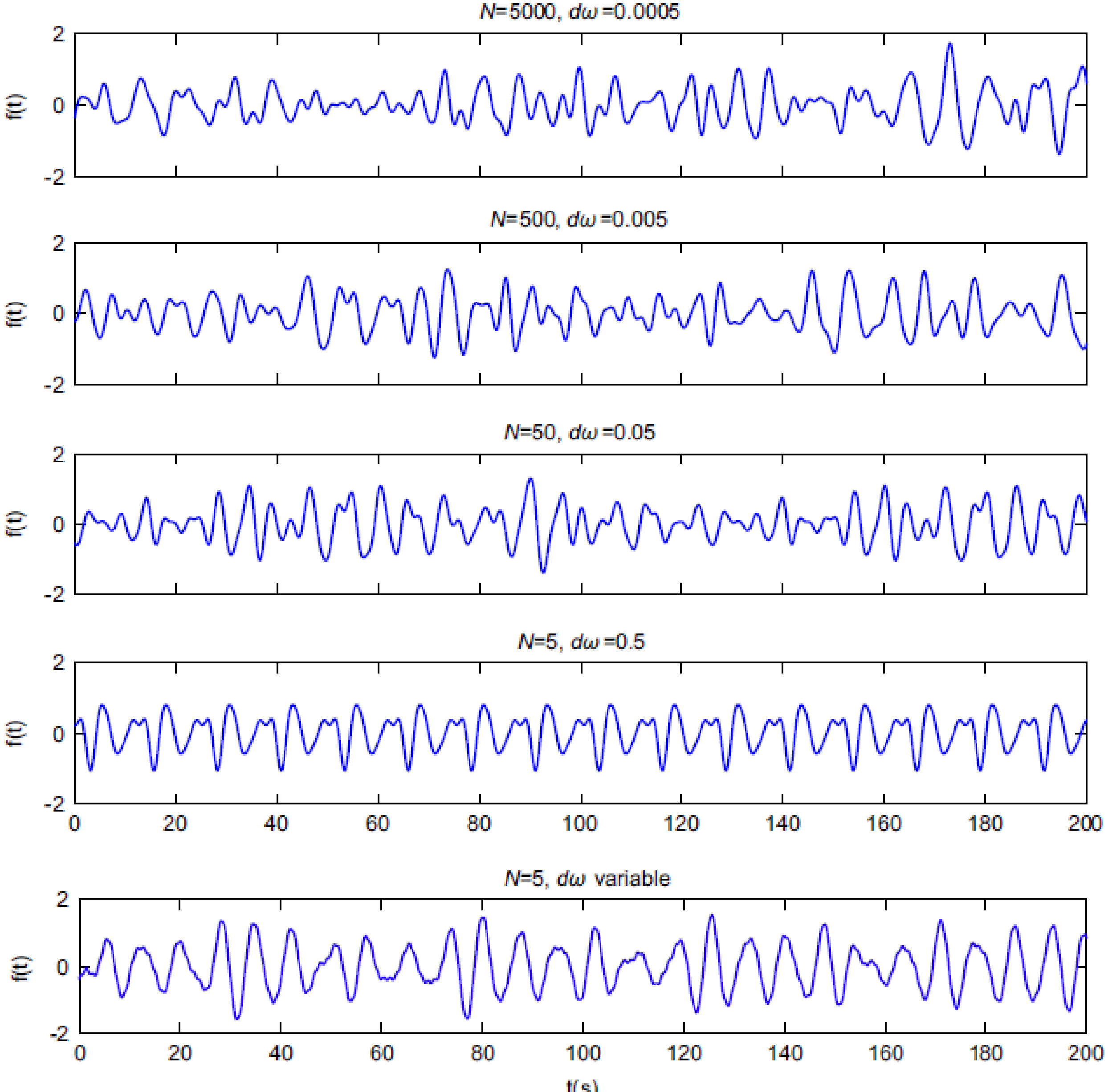
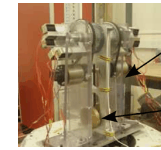
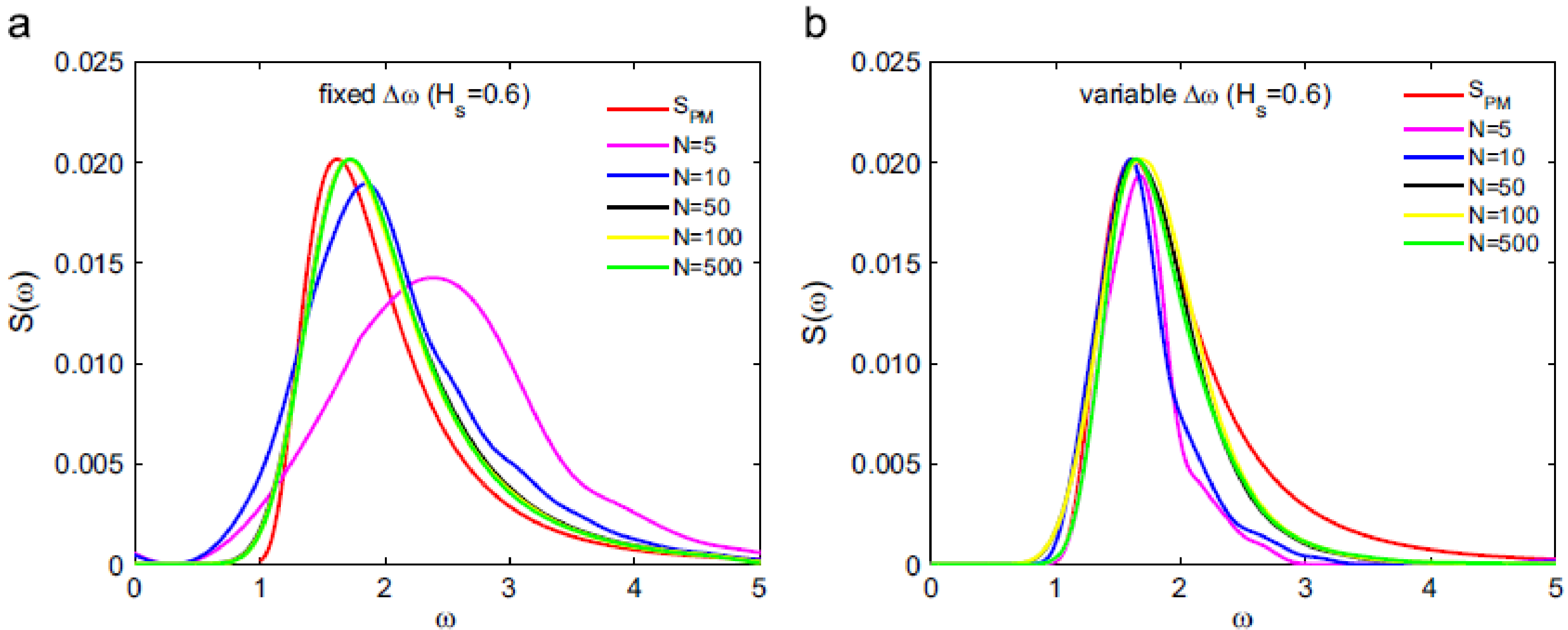
Stochasticity of Excitation





Stochasticity of Excitation





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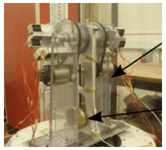
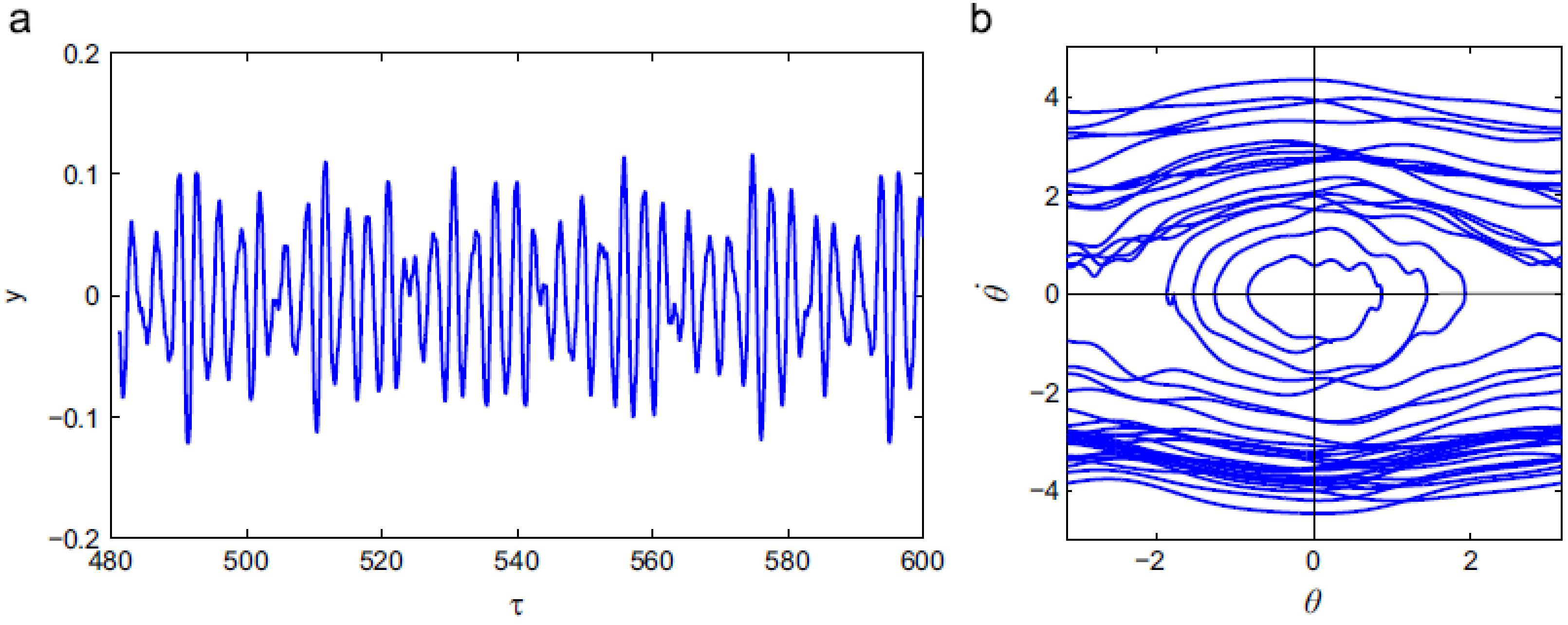
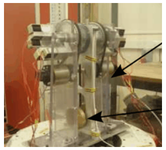
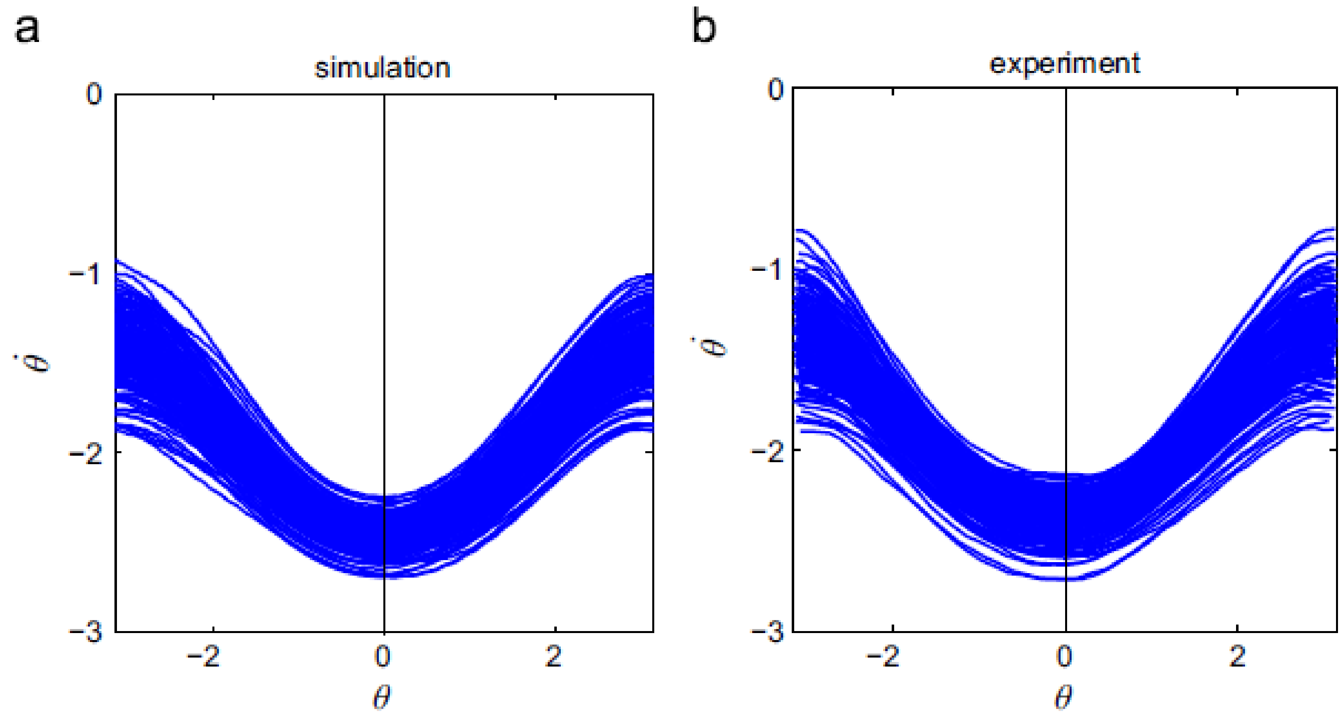


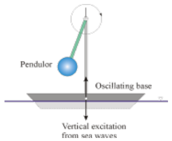


Stochasticity of Excitation

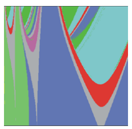




Summary/Closing Remarks



A new concept of energy *extraction* has been shown.

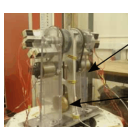


A detailed analysis of a PP has been presented.

Interactions Pendulum vs Shaker have been investigated.

Experiments in a water tank have confirmed the principle.

Bifurcation control and planar excitation has been found beneficial.



Current work is focussed on stochastic excitation and pendula systems.